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GLOBAL CROSS SECTIONS AND INVARIANT MEASURES WITH APPLICATIONS TO THE DENSITIES OF MAXIMAL INVARIANTS

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THESIS

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Uwe Koehn

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To

my wife

and

my parents

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CHAPTER I
INTRODUCTION AND SUMMARY

In considering a parametric testing problem on a sample space X, a subset of euclidean space, one often finds that the problem is invariant under a group of matrix transformations, G, and/or a group of translations. We shall restrict our attention to groups of matrix transformations. (See Lehmann, Testing Statistical Hypotheses [13], chapter 6 for definitions and background.) In these cases the statistician usually requires that the test statistic, say f, also be invariant under the group G, i.e., if $x \in X$, $g \in G$ and gx is the image of x under the transformation g, $f(x) = f(gx)$. Calling the sets of the form $Gx = \{y \in X | y = gx$ for some $g \in G\}$ orbits, f being invariant means that f is constant on orbits.

The common method of attack in this situation is to find a maximal invariant function and to calculate its distribution under the various hypotheses. A maximal invariant function, t, on X is an invariant function that distinguishes between orbits, i.e., if $y \notin Gx$, then $t(x) \neq t(y)$. As a practical matter the range space of t is usually taken to be an open subset of euclidean space which may require the removal of a null set from X. The usefulness of a maximal invariant is that, ignoring measurability problems, any invariant test can be considered as a function of the maximal invariant. The underlying characteristic of a maximal invariant giving us the above property is its association of distinct orbits with distinct points in its range.

In 1956, Stein [14] developed the concept of the calculation of the density of a maximal invariant function by integration over the invariance group. Wijsman [17] at the Fifth Berkeley Symposium presented the derivation of the distributions of some maximal invariants by using the concept of a global cross section along with integration over the invariance group. We seek to generalize some of Wijsman's results. Z, a global cross section of X, is a subset of X that among other properties meets each orbit (almost every orbit will suffice for our needs) in a single point of the orbit. One can, under certain conditions, find the marginal distribution induced on Z by integrating over the group G. Any invariant function on X can be identified with a function on Z, namely its restriction to Z. Since under our conditions Z will turn out to be closed in X, measurability is not a problem.

To be somewhat more specific, let X be a submanifold of E^p and let G be a matrix Lie group acting on X as a Lie transformation group. If $x \in X$, the isotropy subgroup of G at x is the subset of G whose elements leave x fixed, i.e., $gx = x$. This subset is easily seen to be a closed subgroup of G. Now let Z be a submanifold of X intersecting each orbit in one point such that $GZ = \{gz | g \in G, z \in Z\} = X$ and each point of Z has the same isotropy subgroup of G which we denote G_Z. Since G_Z is closed, $G/G_Z = \overline{G}$ has a natural analytic structure making it an analytic manifold. We define the map $f: \overline{G} \times Z \longrightarrow X$ by $f(\overline{g},z) = gz$, $g \in G$, $z \in Z$, where by $\overline{g}$ we shall mean the image of g in $\overline{G}$. Under these conditions we show that f is an analytic homeomorphism from $\overline{G} \times Z^*$ onto X^* where $X - X^*$ is a closed null set in X and Z^* is open in Z.

After showing that a natural measure on Z^* exists, we calculate as examples the induced marginal density on Z^* arising ~~in~~ some multivariate normal problems.

CHAPTER II
NOTATION AND PRELIMINARIES

In this chapter we introduce some definitions, notation and preliminaries. The list of definitions is by no means complete and the reader is referred to P. M. Cohn's Lie Groups [4] and Claude Chevalley's Theory of Lie Groups [3], vol. 1, for the definitions of various concepts from Differential Geometry and Lie Group Theory used in the dissertation but not given in this chapter.

An analytic manifold M is a Hausdorff space with countably many components such that at each point $m \in M$ there is a pair (h,U), U, an open subset of M, $m \in U$, and h, a homeomorphism of U onto an open subset of some euclidean space. Furthermore, if (h,U) and (k,V) are two such pairs on M, then $h \circ k^{-1}$ and $k \circ h^{-1}$ are analytic maps of euclidean space into euclidean space. At times we shall call an analytic manifold merely a manifold. Our definition of an analytic manifold differs in its import in one important respect from that of Chevalley; he requires a manifold to be connected. However, many of his results apply to our manifolds by merely applying them to each component. We make the assumption that the dimension of M is the same at each point of the manifold. If $(u_1,\ldots,u_n)$ are the coordinate functions on euclidean space restricted to $h(U)$, then the functions $(u_1 \circ h, \ldots, u_n \circ h)$ are a local coordinate system at m on $U \subset M$. U is called a coordinate neighborhood, if it has coordinate functions $(u_1 \circ h, \ldots, u_n \circ h)$ defined on it, h is a coordinate map and the system of functions $(u_1 \circ h, \ldots, u_n \circ n)$ is called a chart or coordinate system.

If f is a map from a manifold M to a manifold N and $(x_1, \ldots, x_m)$ is a chart at $m_o \in M$ and $(y_1, \ldots, y_n)$ a chart at $f(m_o) \in N$, then f has an expression in terms of these charts:

$$y_i = f_i(x_1, \ldots, x_m) \qquad i = 1, \ldots, n.$$

f is analytic at m_o if the f_i, $i = 1, \ldots, n$, are analytic functions of the x_j, $j = 1, \ldots, m$. The Jacobian matrix of f at m_o is the matrix, $(\partial f_i / \partial x_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, with the partial derivatives evaluated at m_o. The rank of f at m_o is defined to be the rank of the Jacobian matrix at m_o. If $m = n$, the determinant of the Jacobian matrix, the Jacobian of f, is a meaningful concept. If f is a one-to-one, analytic map of a neighborhood of $m_o \in M$ and the Jacobian is nonzero at m_o, then f^{-1} exists and is an analytic map of a neighborhood of $f(m_o)$ onto a neighborhood of m_o by the Inverse Function theorem (Cohn [4], page 159, Theorem A3). If f is one-to-one, onto, analytic and f^{-1} is analytic, f is called an analytic homeomorphism.

A manifold Z is a submanifold of a manifold M if $Z \subset M$, the identity map is analytic and the Jacobian matrix of the identity map is of maximal rank at each point of Z.

A Lie group G is both a manifold and a group such that the map $(g,h) \longrightarrow gh^{-1}$ is analytic. If H is a Lie group, H is an analytic subgroup of G if H is a subgroup of G that is also a submanifold. If H is a subgroup of G that is closed as a subset of G, then H can be defined as an analytic subgroup of G (Cohn [4], page 123, Theorem 6.5.1). A Lie transformation group is a pair (G,M) where G is a Lie group, M an analytic manifold and G acts as a transformation group on M such that the map $(g,m) \longrightarrow gm$, $g \in G$, $m \in M$ is analytic (Cohn [4], page 66). We shall also call G a Lie transformation group when such a pair (G,M) exists. The subgroup of G that leaves a point $m \in M$ fixed, i.e., $G_m = \{g \in G | gm = m\}$ is called the isotropy subgroup at m. By the continuity of the map $(g,m) \longrightarrow gm$, G_m is closed. If the isotropy subgroup at each point of a set Z is the same, we denote it by G_Z. The coset space, G/G_Z, which we write as $\overline{G}$ or Y, can be given a natural analytic structure such that the map from $G \times Y \longrightarrow Y$, $(g,hG_Z) \longrightarrow ghG_Z$ is analytic. The natural map π: $G \longrightarrow Y$ is both open and continuous. For details see Chevalley [3], pages 110-111. We denote the elements of Y by y or, if we wish to indicate a preimage or the set of preimages in G, by $\overline{g}$ or gG_Z.

If M is a manifold and $(x_1, \ldots, x_m)$ is a chart near $m_o \in M$, then the tangent space at m_o, $T_{m_o}(M)$, has a basis, $((\partial/\partial x_1)_{m_o}, \ldots, (\partial/\partial x_m)_{m_o})$ (Chevalley [3], pages 77-78). The corresponding basis of the dual space is $(dx_1, \ldots, dx_m)$ and an analytic differential form, w_M, of order m has the representation, $f(\ )dx_1 \wedge \ldots \wedge dx_m$, f analytic, in a neighborhood of m_o (Chevalley [3], page 147). If in each such representation f vanishes at no point and w_M is defined on all M, w_M is a nonzero analytic differential form on M, for short, a differential form. If h is an analytic homeomorphism from M to N, then $(x_1 \circ h^{-1}, \ldots, x_m \circ h^{-1})$ is a chart on N near $h(m_o)$ and $(d(x_1 \circ h^{-1}), \ldots, d(x_m \circ h^{-1}))$ is a dual basis to $T_{h(m_o)}(N)$, the tangent space of N at $h(m_o)$.
$f \circ h^{-1}(\ )d(x_1 \circ h^{-1}) \wedge \ldots \wedge d(x_m \circ h^{-1})$ is a representation near $h(m_o)$ of the differential form, $\delta h^{-1}(w_M)$, on N. In cases where such an analytic homeomorphism exists, we shall often identify the two spaces and their differential forms. Similarly, in the construction of product manifolds, differential forms are mapped onto the product manifold from the factor manifolds and we shall identify the differential form

on the product manifold (the differential form is in general not of degree equal to the dimension of the product manifold) with its preimage differential form on the factor manifold. (Chevalley [3], pages 165-7.)

For the purposes of this dissertation all manifolds are assumed to be submanifolds of euclidean space, E^p, and all Lie groups are analytic subgroups of the general linear group, $GL(p,R)$. Furthermore the action of a Lie group on a manifold is assumed to be that of a group of linear transformations. This is possible since the group elements have representations as matrices and the points of the manifold as vectors in E^p. It is not assumed that the manifold is a vector subspace of E^p. Since our Lie groups are subgroups of $GL(p,R)$ and our manifolds are submanifolds of E^p, the group product, gh, also denotes matrix multiplication and the group action on a manifold, gm, the multiplication of a vector by a matrix.

Y, the coset space of G with respect to some closed subgroup of G, is not in general a submanifold of G as can be seen by considering the additive reals modulo the additive integers. However, by the construction of Y (Chevalley [3], pages 110-111), there is a neighborhood of $\pi(e)$, the image of the identity, that is analytically homeomorphic to a submanifold of G. By translation by the elements of G, the above sentence implies that every point of Y has a neighborhood analytically homeomorphic to a submanifold of G.

Using Chevalley's [3] proposition 2, page 96, we see that our manifolds, including G, have a countable base for their topologies. Y also has a countable base since the natural map, π, is both open and continuous.

As is shown in Chevalley [3], pages 161-164, if a manifold has a nonzero analytic differential form of degree equal to the dimension of the manifold, one can define an integral on the manifold for continuous functions with compact support. A Lie group always has such a differential form which generates an integral that is invariant under left group translations. This differential form is unique up to a multiplicative constant. If G is a Lie group, a left invariant differential form on G is denoted w_G. A differential form of maximal degree on a manifold M is denoted w_M. By means of these forms, measures can be created (see Appendix).

We shall also use the properties of the exponential map on G (Chevalley [3], Pages 115 ff. and Cohn [4], pages 76 and 107ff.).

CHAPTER III
SARD'S THEOREM AND A DIMENSION LEMMA

Let f be an analytic map from M_1 to M_2, two manifolds of dimensions n_1 and n_2, respectively. The set of points in M_1 at which the rank of f is less than n_2 are called the critical points of f. All other points in M_1 are called regular. A point, q, of M_2 such that $f^{-1}(q)$ contains at least one critical point is called a critical value. All other points of M_2 are regular values. From these definitions, it is clear that if $n_1 < n_2$, every point of M_1 is a critical point and $f(M_1)$ is a set of critical values.

We need a definition of measure zero on manifolds. A set $A \subset M$, a manifold, is of measure zero if for any coordinate neighborhood U and its coordinate map h into euclidean space, $h(A \cap U)$ has Lebesgue measure zero.

The following is a version of Sard's theorem restricted to our needs:

THEOREM 1. Let f be an analytic map from the manifold M_1 to the manifold M_2. The set of critical values of f is a set of measure zero in M_2.

A good reference for Sard's theorem is S. Sternberg's <u>Lectures on Differential Geometry</u> [15], pages 45-55.

The following lemma is useful.

LEMMA 1. <u>If</u> f <u>is a one-to-one, onto, analytic map from</u> M_1, <u>a manifold, to</u> M_2, <u>a manifold, then the dimension of</u> M_1 <u>equals the dimension of</u> M_2.

Proof. First we show that $\dim(M_1) \geq \dim(M_2)$. If not, $\dim(M_1) < \dim(M_2)$. But then the rank of f is less than $\dim(M_2)$ and all points of M_2 are critical values since f is onto. Thus, by Sard's theorem, M_2 is a null set. Since $\dim(M_2) \geq 1$, this is a contradiction.

Since $\dim(M_1) \geq \dim(M_2)$, by Sard's theorem there is at least one point, say $p \in M_1$, such that f has rank equal to $\dim(M_2)$, which implies that df, the tangent map, is onto at $f(p)$ (Chevalley [3], pages 78-80). By proposition 2, page 80, in Chevalley [3], if $(y_1, \ldots, y_n)$, $n = \dim(M_2)$, is a chart near $f(p)$, then there exists a chart $(y_1 \circ f, \ldots, y_n \circ f,\ z_{n+1}, \ldots, z_{n+r})$ near p in M_1. Clearly in the neighborhood, say
$N = \{q \in M_1 \mid |(y_i \circ f)(q)| < \epsilon \quad i = 1, \ldots, n;\ |z_j(q)| < \epsilon \quad j = n+1, \ldots, n+r\}$, f is an open map. But then f is a homeomorphism near p and so $\dim(M_1) = \dim(M_2)$.

CHAPTER IV
ANALYTIC HOMEOMORPHISM THEOREM

We now prove the main result. Let G be an analytic subgroup of the general linear group, $GL(p,R)$, acting as a Lie transformation group on X, a submanifold of E^p. Let Z be a submanifold of X. If $x \in X$, the elements of G that leave x fixed, i.e., G_x, form a closed subgroup of G since the map, $G \times X \longrightarrow X$, is analytic and so continuous.

THEOREM 2. <u>If the following conditions hold</u>:

(1) if $z_1, z_2 \in Z$, then $G_{z_1} = G_{z_2}$

(we denote this common subgroup G_Z)

(2) if $z_1, z_2 \in Z$ and $gz_1 = z_2$, then $g \in G_Z$ (and so $z_1 = z_2$)

(3) $GZ = X$

<u>then there exists an open submanifold</u>, Z^*, <u>of</u> Z <u>and an open submanifold of</u> X, <u>called</u> X^*, <u>such that</u> $X - X^*$ <u>is of measure zero (as defined above) and the map</u>, $f\colon Y \times Z \longrightarrow X$, <u>defined by</u> $f(\bar{g}, z) = gz$, <u>restricted to</u> $Y \times Z^*$, <u>is an analytic homeomorphism onto</u> X^*, <u>where</u> $Y = G/G_Z$.

Proof. f is well defined since, if $g_1 = g_2 g^*$, $g^* \in G_Z$, then $g_1 z = g_2 g^* z = g_2 z$, $z \in Z$. It is also one-to-one since, if $g_1 z_1 = g_2 z_2$, then by condition (2), $g_1 \in g_2 G_Z$ and $z_1 = z_2$. By condition (3), f is onto.

Let $(L_{r+1},\ldots,L_s)$ be a basis for the tangent space of G_Z at e, $T_e(G_Z)$, and $(L_1,\ldots,L_r)$ a complementary set in the tangent space of G at e, the identity. $(L_1,\ldots,L_s)$ is then a basis for $T_e(G)$. By this choice $(L_1,\ldots,L_r)$ is a basis for the tangent space of Y at $\bar{e}$, $T_{\bar{e}}(Y)$. Using this basis, $g \exp \sum_{i=1}^{r} a_i L_i \cdot \exp \sum_{i=r+1}^{s} a_i L_i \longrightarrow$

$(a_1,\ldots,a_r, a_{r+1},\ldots,a_s)$ is a canonical chart near $g \in G$ and $g \exp \sum_{i=1}^{r} a_i L_i \longrightarrow (a_1,\ldots,a_r)$ is a canonical chart

near $\bar{g} \in Y$. Since X is a submanifold of E^p, Z is also and so the elements of Z have coordinates $(z_1(t),\ldots,z_p(t))$ in terms of some orthonormal coordinate system in E^p, where $(t) = (t_1,\ldots,t_q)$ is a chart on some open set in Z. The $z_i(t)$ are analytic functions of (t), since Z is a submanifold of E^p. Under the map from $G \times Z$ onto X given by $(g,z) \longrightarrow gz$, the point (a,t) maps into

$$g \exp \sum_1^r a_i L_i \cdot \exp \sum_{r+1}^{s} a_i L_i (z_1(t),\ldots,z_p(t))' = g \exp \sum_1^r a_i L_i \cdot z(t),$$

since $\exp \sum_{r+1}^{s} a_i L_i \in G_Z$ and where $z(t) = (z_1(t),\ldots,z_p(t))'$.

(We are considering the elements of G as matrices, the elements of X as column vectors, and prime denotes transposition.) From the above we see that $f\colon Y \times Z \longrightarrow X$ is analytic.

Thus f is a one-to-one, onto, analytic map of $Y \times Z$ onto X and, by lemma 1, the dimension of $Y \times Z$ equals the dimension of X. Call this common dimension n. Furthermore, by Sard's theorem, f has rank n at almost every range value in X. The set of points, A, in $Y \times Z$ at which the rank of f is n, is an open set since it is the set on which the Jacobian of f, a continuous function, is nonzero. By the Inverse Function theorem (Cohn [4], page 159, Theorem A3), f is an analytic homeomorphism on A and so $f(A)$, which we denote X^*, is open in X. $X - X^*$ is then the set of measure zero mentioned in Sard's theorem.

To see the nature of A and X^* more clearly we note that f is equivariant under the actions of G, i.e., $f(ghG_Z,z) = ghz = gf(hG_Z,z)$, $g,h \in G$. Since the action of any element $g \in G$ on Y is an analytic homeomorphism (Chevalley [3], page 111), the action $g\colon(y,z) \longrightarrow (gy,z)$ on $Y \times Z$ is an analytic homeomorphism. If f is an analytic homeomorphism on a neighborhood of a point $(y_1,z_1) \in Y \times Z$, then it is one at each point (y_2,z_1), z_1 fixed, $y_2 \in Y$. This fact is shown by letting $g \in G$ be chosen such that

$y_2 = gy_1$. Then the map f near (y_2, z_1) can be written as the composition map,

$$(y,z) \xrightarrow{g^{-1}} (g^{-1}y,z) \xrightarrow{f} f(g^{-1}y,z) = g^{-1}f(y,z) \xrightarrow{g} f(y,z),$$

where each of the maps is an analytic homeomorphism, the last since (G,X) is a Lie transformation group.

Thus A has the form $Y \times Z^*$ where Z^* is open in Z since $Y \times Z^*$ is open and the projection map is an open map. X^* is a union of orbits. q.e.d.

Since $X - X^*$ is a null set, we shall remove it from consideration in any future work and write X for X^* and Z for Z^*.

The above theorem gives us f as an analytic homeomorphism at almost all points of X. Wijsman [17] has shown that if Z is generated by a Lie transformation group, H, of X, i.e., $Hx_o = Z$, $x_o \in X$, and H either commutes with G in GH or is normal in GH where GH is a transitive Lie transformation group of X, then the map $L: \overline{G} \times H/H_{x_o} \longrightarrow X$ defined by $L(\overline{g},\overline{h}) = ghx_o$ is an analytic homeomorphism of $\overline{G} \times H/H_{x_o}$ onto X under conditions corresponding to those of the above theorem. The question arises whether or not L is, in general, an analytic homeomorphism onto all of X, i.e., everywhere, instead of just almost everywhere. Or, if not, whether it is one if we can generate Z by a group H when $GH = \{gh \mid g \in G,\ h \in H\}$ is not a group. If GH is a group, it can be shown that L is an analytic homeomorphism everywhere.

The following example shows that the almost everywhere statement in theorem 2 is necessary even in the case that Z is generated by a Lie transformation group H on X if GH is not a group. Let X be the plane,

$\{(1,y,z)' \mid -\infty < y < \infty,\ -\infty < z < \infty\}$, in E^3 and

$G = \{e^{sK} \mid K = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\} = \{g \mid g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ s^2/2 & s & 1 \end{pmatrix}\}$. We choose

$x_o = (1,0,0)'$ and $H = \{e^{tN} \mid N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},\ -\infty < t < \infty\}$. An

element of H has the form, $\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ e^t-t-1, & e^t-1, & e^t \end{pmatrix}$, and so

$Z = Hx_o = \{(1,t,e^t-t-1)' \mid -\infty < t < \infty\}$. Clearly H_{x_o} is the trivial group. G and H are easily seen to be Lie transformation groups on X.

To check conditions (1) and (2) of theorem 2, let $e^{sK}e^{tN}x_o = e^{uN}x_o$ for some s, t and u. In terms of these coordinates, we have

$$u = s + t$$

$$e^u - u - 1 = (s^2/2) + ts + e^t - t - 1.$$

If $t = u$, then $s = 0$ and $e^{0K} = I$. Thus condition (1) is verified with $G_Z = \{I\}$. If $t \neq u$, $s = u - t \neq 0$ and

$$\frac{(u-t)^2}{2} + t(u-t) + e^t - t = e^u - u$$

implying that

$$\frac{u + t}{2} + 1 = \frac{e^u - e^t}{u - t} .$$

Letting $u - t = 2v \neq 0$ and $u + t = 2r$, we have

$$(1) \qquad \frac{r + 1}{e^r} = \frac{\sinh (v)}{v} .$$

By the Law of the Mean, $\frac{\sinh (v)}{v} = \cosh (v^*)$, v^* in the open interval bounded by 0 and v, so $\cosh (v^*) > 1$. Since the exponential function is convex and $y = r + 1$ is the tangent line to it at $r = 0$, $\frac{r + 1}{e^r} \leq 1$ and (1) cannot hold for any r and v, $v \neq 0$. Thus we have a contradiction and condition (2) is verified.

To verify condition (3), observe that as functions of s and t

$$y = s + t$$

$$z = (s^2/2) + ts + e^t - t - 1.$$

Clearly the range of y is $(-\infty,\infty)$. Let $u = s + t$, $t = t$ be a change of coordinates so that

$$y(u,t) = u$$

$$z(u,t) = \tfrac{1}{2}u^2 + e^t - (\tfrac{1}{2}t^2 + t + 1).$$

For fixed, arbitrary u, as $t \longrightarrow \infty$, e^t dominates $-(\frac{1}{2}t^2 + t + 1)$ and so $z(u,t) \longrightarrow \infty$. As $t \longrightarrow \infty$, $e^t \longrightarrow 0$ and so $z(u,t) \longrightarrow -\infty$. Thus $GHx_o = X$.

So theorem 2 holds and we have an analytic homeomorphism almost everywhere in X. It is not one everywhere, since at $(s,t) = (0,0)$, $\frac{d}{ds} e^{sK}x_o = Kx_o = (0,1,0)'$ and $\frac{d}{dt} e^{tN}x_o = Nx_o = (0,1,0)'$.

In this example, Z could be chosen to be $\{(1,0,t)' \mid -\infty < t < \infty\}$ and we would have an analytic homeomorphism everywhere. Whether this choice is always possible seems to be unknown.

CHAPTER V
MEASURES AND DIFFERENTIAL FORMS

In our applications X is an orientable analytic manifold with a non-zero differential form of maximal order (Chevalley [3], page 158ff.). This differential form on X generates a Baire measure on X as is shown in the Appendix. Since we will be "factoring" X into the product of two analytic manifolds, Y and Z, we would like to have measures on Y and Z such that the product measure on $Y \times Z$ is equivalent to the measure on X. A method of calculating the Radon-Nikodym derivative is also desirable. If G_Z is compact, Y has a natural measure inherited from G, the Lie group, in the following way: let μ be "the" Haar measure on G and $\pi\colon G \longrightarrow Y$, the natural map; we define $\mu_Y = \mu\pi^{-1}$ on Y by $\mu_Y(A) = \mu\pi^{-1}(A) = \mu(\pi^{-1}(A))$, A being a Baire set of Y. μ_Y is a regular measure (Halmos [7], Theorem G, page 228), invariant under the actions of G and unique up to a multiplicative constant (Helgason [8], Theorem 1.7, page 369). Although Haar measure and our other invariant measures are unique only up to a multiplicative constant, we shall use the definite article in referring to them. If a measure is invariant under the actions of a group G, we shall call it G-invariant.

In general, even though Y is an analytic manifold, μ_Y is not induced by a G-invariant differential form. In fact, even the coset space of a compact Lie group modulo a compact subgroup need not be orientable, e.g., $SO(3)/O(2)$ = the projective plane (Helgason [8], page 369). On reading chapter X of Helgason's Differential Geometry and Symmetric Spaces [8], one suspects that the problem is that π, the natural map, maps analytically homeomorphic sets in G onto the same set in Y in such a way that orientations are different. If this is the case, we can hope to put a differential form on open subsets of Y by restricting the domain of π.

First we shall show that we can restrict our attention to G_o, the component of the identity, e, of G. For ease of notation, let us denote the compact subgroup of G (the isotropy subgroup of theorem 2) by H and let $H_1 = G_o \cap H$. H_1 is a compact subgroup of G_o and so of G. π is the natural map, $\pi: G \longrightarrow G/H$ and π_1, the natural map, $\pi_1: G_o \longrightarrow G_o/H_1$. π and π_1 are open and continuous. The set G_oH is an open subgroup of G, since G_o is open and a normal subgroup of G (Cohn [4], Theorem 2.4.1 and Theorem 2.8.3). If π is restricted to G_o, the image $\pi(G_o) = G_oH/H$ and the preimage of $\pi(g)$, $g \in G_o$, is gH_1. Thus, since π_1 partitions G_o in the same way, there exists a map f such that $f: G_oH/H \longrightarrow G_o/H$, $f \circ \pi|_{G_o} = \pi_1$ and

$\pi|_{G_o} = f^{-1} \circ \pi_1$. Since $\pi|_{G_o}$ and π_1 are continuous and open, f is a homeomorphism. If μ is the Haar measure on G, its restrictions to the open subgroups G_oH and G_o are also the Haar measures for these groups. Let us denote the G_oH-invariant measure induced on G_oH/H by μ_2 and the G_o-invariant measure on G_o/H_1 by μ_1. Because of the équivariance of f under G_o, i.e., $gf(x) = f(gx)$, $g \in G_o$, $x \in G_oH/H$, $\mu_2 f^{-1}$ and μ_1 are G_o-invariant measures on G_o/H_1 and so $\mu_2 f^{-1} = k\mu_1$ due to the uniqueness of the invariant measure. But then $k\mu_1 f = \mu_2 f^{-1} f = \mu_2$ and so we have a one-to-one correspondence between the G_o-invariant measures on G_o/H_1 and the G_oH-invariant measures on G_oH/H. μ_2 is the restriction of the G-invariant measure on G/H to G_oH/H, an open subset. Because of the G-invariance and second countability, μ_2 determines the measure of all measurable sets of G/H. To see this fact, let A be a measurable set and let $\bigcup_{i=1}^{\infty} g_i G_o H/H \supset A$. Let $A_1 = A \cap g_1 G_o H/H$ and $A_i = A \cap g_i G_o H/H - \bigcup_{k=1}^{i-1} A_k$ so that $A_i \cap A_j = \phi$, $i \neq j$, and $A = \bigcup A_i$. Then $\mu\pi^{-1}(A) = \mu\pi^{-1}(\bigcup A_i) = \Sigma\mu\pi^{-1}(A_i) = \Sigma\mu\pi^{-1}(g_i^{-1}A_i) = \Sigma\mu_2(g_i^{-1}A_i)$, since $g_i^{-1}A_i \subset G_oH/H$.

Now assume that G is connected. We shall show that the G-invariant measure on G/H is generated in an open neighborhood of $\pi(e)$ by a differential form. This fact will allow us to use differential forms to obtain relationships between measures.

Let H_o be the identity component of H and $\pi_o\colon G \longrightarrow G/H_o$, the natural map. Being a closed subset of a compact set, H_o is compact in H and G. Since H_o is connected, Helgason's [8], lemma 1.5 and proposition 1.6, pages 367-8, show that G/H_o has a G-invariant differential form, w, that generates the G-invariant measure, $\mu\pi_o^{-1}$.

To find a differential form that generates the invariant measure even on an open subset of G/H and its relation to w on G/H_o requires more work. Let θ be the map, $\theta\colon G/H_o \longrightarrow G/H$, $\theta(gH_o) = gH$, $g \in G$, so that $\pi = \theta \circ \pi_o$. In the following we shall follow to the extent possible the notation of Chevalley [3], page 58, proposition 4. We can choose an open, connected neighborhood V of e such that $V^{-1}V \cap H \subset H_o$. Let $W = \pi(V)$. Choose a collection Δ of distinct representatives of H_o cosets in H so that we can write $H = \sum_{\delta \in \Delta} \delta H_o$, where addition means disjoint union. Since H is compact, Δ is a finite collection, so we write $H = \sum_{i=1}^{k} \delta_i H_o$, $\delta_1 = e$. In the proof of proposition 4, Chevalley shows that the sets $U_i = \pi_o(V\delta_i)$ are disjoint

so that $\theta^{-1}(W) = \pi_o\pi^{-1}(W) = \pi_o(VH) = \pi_o(V \Sigma \delta_i H_o) =$
$\pi_o(\Sigma V\delta_i H_o) = \Sigma \pi_o(V\delta_i H_o)$ or

$$(2) \qquad \theta^{-1}(W) = \Sigma\pi_o(V\delta_i) = \Sigma U_i .$$

Chevalley shows that the U_i are components of $\theta^{-1}(W)$. $\pi(e)$ is an element of the component U_1, which for notational convenience we write U. Chevalley's [3], proposition 4 states that $\theta|_{U_i} : U_i \longrightarrow W$ is a homeomorphism. It is useful to show that it is an analytic homeomorphism and we shall do this for U. Let $(L_1,\ldots,L_n)$ be a basis for the tangent space of G at e such that $(L_{m+1},\ldots,L_n)$ is a basis for the tangent space of H (and so H_o) at e. We can assume that V is small enough so that

$$\exp \sum_{i=1}^{m} a_i L_i \ \exp \sum_{m+1}^{n} a_i L_i \longrightarrow (a_1,\ldots,a_n)$$

is a chart on V. Then

$$\pi_o(\exp \sum_{i=1}^{m} a_i L_i \ \exp \sum_{m+1}^{n} a_i L_i) \longrightarrow (a_1,\ldots,a_m)$$

is a chart on U and

$$\pi(\exp \sum_{i=1}^{m} b_i L_i \ \exp \sum_{m+1}^{n} b_i L_i) \longrightarrow (b_1,\ldots,b_m)$$

is a chart on W. In terms of these charts, $b_i = \theta^i|_U (a_1,\ldots,a_m)$, $i = 1,\ldots,m$, and, by definition of θ, $b_i = a_i$ so that in terms of these charts, $\theta|_U$ is the identity function and so analytic.

Denoting the analytic homeomorphism $\theta|_U^{-1}$ by φ, we define the differential form w_φ on W by $w_\varphi = \delta\varphi(w)$. We want to exhibit the relation between w_φ and $\mu\pi^{-1}$. Let D be a measurable subset of W and let $V_D = V \cap \pi^{-1}(D)$. Then $\pi(V_D) = D$ since $\pi(V \cap \pi^{-1}(D)) \subset \pi\pi^{-1}(D) = D$ and since for any $d \in D$, there is a $g \in V$ such that $\pi(g) = d(\pi(V) = W \supset D)$, implying that $g \in V \cap \pi^{-1}(D)$ so that $\pi(V \cap \pi^{-1}(D)) \supset D$. Replacing in (2) V by V_D and W by D (only the property $V^{-1}V \cap H \subset H_o$ is used to establish (2) and V_D enjoys the same property), we get

$$(3) \qquad \theta^{-1}(D) = \Sigma\, \pi_o(V_D\delta_i)$$

and so $\pi^{-1}(D) = \pi_o{}^{-1}\theta^{-1}(D) = \Sigma\, V_D\delta_i H_o$. Thus $\mu(\pi^{-1}(D)) = \Sigma\, \mu(V_D\delta_i H_o) = \Sigma\, \mu(V_D H_o \delta_i) = k\mu(V_D H_o) = k\mu\pi_o{}^{-1}(\pi_o(V_D))$, the second equality occurring because $\delta_i \in H$ and H_o is normal in H and the third because H is compact and therefore the modular function is equal to 1. In order to write $\pi_o(V_D)$ more conveniently, we observe that the i-th term on the right in (3), $\pi_o(V_D\delta_i) \subset \pi_o(V\delta_i) = U_i$. Therefore,

$\theta^{-1}(D) \cap U_i = \pi_o(V_D\delta_i)$ and, in particular, $\pi_o(V_D) = \theta^{-1}(D) \cap U = \varphi(D)$, $(\varphi = \theta|_U^{-1})$. Thus we get

$$(4) \qquad \mu\pi^{-1} = k\mu\pi_o^{-1}\varphi.$$

Since w generates $\mu\pi_o^{-1}$, $w_\varphi = \delta\varphi(w)$ and if μ_φ is the measure generated by w_φ on W, then by (33) of the Appendix, it follows that $\mu_\varphi = \mu\pi_o^{-1}\varphi$. Comparing this equation with (4) yields

$$(5) \qquad \mu\pi^{-1} = k\mu\varphi.$$

Thus kw_φ generates the G-invariant measure, $\mu\pi^{-1}$, on W. In the future we shall write w_φ for kw_φ. Due to the G-invariance, the measure of the subsets of W determines the $\mu\pi^{-1}$ measure of all measurable sets in G/H. By the results shown earlier, the measure generated on W by w_φ can be extended uniquely by G-invariance to all G/H even if G is not connected. We summarize these results in a theorem.

THEOREM 3. Let G be a Lie group with countably many components and Haar measure, μ, and let H be a compact subgroup of G with π being the natural map, $\pi: G \longrightarrow G/H$.

There exists an open neighborhood of $\pi(e) \in G/H$ *on which the G-invariant measure,* $\mu\pi^{-1}$*, on* G/H *is generated by a differential form.*

Returning to the situation where $Y \times Z$ is analytically homeomorphic to X, we find that the existence of a differential form on an open neighborhood of $\pi(e)$ implies that Z has a nonzero differential form of maximal order by the following lemma.

LEMMA 2. *Let* U, V *and* W *be analytic manifolds such that* $U \times V$ *is analytically homeomorphic to* W. *Furthermore let there exist differential forms of maximal degree,* w_U *and* w_W*, on* U *and* W*, respectively. Then there exists a differential form,* w_V*, on* V *such that*

$$w_W = \delta(\cdot,\cdot)w_U \wedge w_V$$

where δ *is analytic and* $\delta > 0$.

Note. We shall denote the images of w_U and w_V on $U \times V$ by w_U and w_V also.

Proof. Let the dimensions of U, V and W be n, m and p, respectively, so that $n + m = p$. Let u_o be an arbitrary but fixed point of U with $(u_1,\ldots,u_n)$ a chart around u_o.

Then

$$w_U(\cdot) = f(\cdot)\, du_1 \wedge \ldots \wedge du_n$$

in terms of this chart with f analytic near u_o. If v_o is a point of V, let $(v_1,\ldots,v_m)$ be a chart in a neighborhood of v_o. Then $(u_1,\ldots,u_n, v_1,\ldots,v_m)$ is a chart in a neighborhood of $(u_o,v_o) \in W$ and

$$w_W(\cdot,\cdot) = h(\cdot,\cdot)\, du_1 \wedge \ldots \wedge du_n \wedge dv_1 \wedge \ldots \wedge dv_m$$

in terms of this chart. Let us define a differential form on V in a neighborhood of v_o by

$$w_V(\cdot) = h(u_o,\cdot)\, dv_1 \wedge \ldots \wedge dv_n.$$

Since $h(u_o,\cdot)$ is analytic and nonzero by the properties of w_W, w_V is a nonzero analytic differential form near v_o. We define w_V on all V by piecing together the above forms defined in a neighborhood of each point of V. To see that this definition is consistent, let $(v_1,\ldots,v_m)$ and $(v_1',\ldots,v_m')$ be two charts whose domains of definition overlap on V and

$$dv_1 \wedge \ldots \wedge dv_m = J(\cdot)\, dv_1' \wedge \ldots \wedge dv_m'$$

where J is the Jacobian of the transformation from the coordinates $(v_1',\ldots,v_m')$ to the coordinates $(v_1,\ldots,v_n)$.

Since we have a change of coordinates, $J \neq 0$ in the neighborhood of overlap. The corresponding change of coordinates in W from $(u_1,\ldots,u_n, v_1',\ldots,v_m')$ to $(u_1,\ldots,u_n, v_1,\ldots,v_m)$ gives us

$$(6)\ du_1 \wedge\ldots\wedge du_n \wedge dv_1 \wedge\ldots\wedge dv_m = $$
$$J(\cdot,\cdot)\ du_1 \wedge\ldots\wedge du_n \wedge dv_1' \wedge\ldots\wedge dv_m'$$

where J can be and is identified with the Jacobian given above. Now

$$w_W(\cdot,\cdot) = h_1(\cdot,\cdot)\ du_1 \wedge \ldots \wedge du_n \wedge dv_1 \wedge \ldots \wedge dv_m$$

and $w_W(\cdot,\cdot) = h_2(\cdot,\cdot)\ du_1 \wedge \ldots \wedge du_n \wedge dv_1' \wedge \ldots \wedge dv_m'$. By (6), $h_1(\cdot,\cdot)J(\cdot) = h_2(\cdot,\cdot)$ and so, in particular, $h_1(u_o,\cdot)J(\cdot) = h_2(u_o,\cdot)$ giving

$$w_V(\cdot) = h_1(u_o,\cdot)dv_1 \wedge \ldots \wedge dv_m = h_1(u_o,\cdot)J(\cdot)dv_1' \wedge \ldots \wedge dv_m'$$
$$= h_2(u_o,\cdot)\ dv_1' \wedge \ldots \wedge dv_m'.$$

Thus the definition of w_V on all V is consistent.

$w_U \wedge w_V$ and w_W are both nonzero analytic differential forms on W. Since at each point these forms of maximal degree are a one dimensional vector space, $w_W = \delta(\cdot,\cdot)\ w_U \wedge w_V$ with δ being analytic since w_W and

$w_U \wedge w_V$ are. By possibly changing the sign of w_V on some or all components of V, since δ is either positive or negative on each component, δ being continuous, we can choose δ to be positive on all W and the lemma is proved.

Since Z now has a nonzero differential form, the lemma also implies that Y has a nonzero differential form and so is orientable. Since the action of any element of G on Y is an analytic homeomorphism and $\mu\pi^{-1}$ is G-invariant, it is tedious but straightforward to show that if $g \in G$, $\delta g^{-1}(w_\theta)$ generates $\mu\pi^{-1}$ on the open set $gV \subset Y$.

Let w_Z and w_X be the differential forms on Z and X, respectively, and u_Z and u_X, the respective measures that they generate. On the set, $gV \times Z$, we have $w_X = f(y,z)\delta g^{-1}(w_\theta) \wedge w_Z$, where $f(y,z) \not\leq 0$ and, since f is analytic and so continuous, f is either less than zero or greater than zero on $gV \times Z$. Since $-w_X$ and w_X generate the same measure, we have $\mu_X = |f(y,z)|\ \mu\pi^{-1} \circ \mu_Z$ at any point $(y,z) \in Y \times Z$. Summarizing the above results, we have

THEOREM 4. Let G be a Lie transformation group acting on an analytic manifold X. Let Z be a submanifold of X for which H is a compact isotropy subgroup of G at each point and such that $Y \times Z$ is analytically homeomorphic to X, where $Y = G/H$. Then, if X has a nonzero differential form, so does Z and the invariant measure on Y is generated by a nonzero differential form in the neighborhood of each point. Furthermore the measures generated by these differential forms satisfy the relationship,

$$\mu_X = f(y,z)\ \mu\pi^{-1} \circ \mu_Z \tag{7}$$

where f is analytic and positive and $\pi: G \longrightarrow Y$ is the natural map.

If, as is often the case, X is an open subset of euclidean space and since G acts as a group of matrices on X, we can say even more. Let μ_X be Lebesgue measure. Due to the G-invariance of $\mu\pi^{-1} = \mu_Y$, if, from (7),

$$\mu_X(dx) = f(y,z)\ \mu_Y(dy)\ \mu_Z(dz)$$

then $\mu_X(gdx) = f(gh,z)\ \mu_Y(gdy)\ \mu_Z(dz)$ and so $|g|\mu_X(dx) = f(gy,z)\ \mu_Y(dy)\ \mu_Z(dz)$ where $|g|$ is the absolute value of the Jacobian of the linear map $g: X \longrightarrow X$. Thus $|g|f(y,z) = f(gy,z)$ and letting $y = \pi(e) = \overline{e}$ and $\pi(g) = \overline{g}$, $|g|f(\overline{e},z) = f(\overline{g},z)$. The choice of the representative of $\pi^{-1}(\overline{g})$ is immaterial since $g_1 \in g_2H$ implies that

$|g_1| = |g_2|$, H being compact. We shall write $f(z)$ for $f(\bar{e},z)$ in these situations. By the above argument we can say that

$$(8) \qquad \mu_X(dx) = |g| f(z)\, \mu_Y(dy)\, \mu_Z(dz).$$

Since $w_X = \pm f(z) w_\varphi \wedge w_Z$, at the points $(\bar{e},z) \in Y \times Z$, we can use differential forms and Jacobians to evaluate $f(z)$ as w_X, w_φ and w_Z are usually known at $(\bar{e},z)$.

Returning to the more general situation, since μ_X, μ_Y and μ_Z are measures on the Baire sets of X, Y and Z respectively, if p is a Baire measurable function on X,

$$(9) \qquad \int_X p(x)\mu_X(dx) = \int_Z \int_Y p(y,z) f(y,z)\mu_Y(dy)\mu_Z(dz)$$

where $p(x) = p(y,z)$, (y,z) being the image of x under the analytic homeomorphism from X to $Y \times Z$. If h is a measurable function from Y to R and $\int h\mu_Y(dy)$ exists, then $\int h\mu_Y(dy) = \int_G h \circ \pi\mu(dg)$ (Lehmann [13], page 38, Lemma 2). For simplicity we shall write h for $h \circ \pi$. As a result, (9) can be written

$$(10) \qquad \int_X p(x)\mu_X(dx) = \int_Z \int_G p(gz) f(g,z)\mu(dg)\mu_Z(dz)$$

where $gz = x$ and $f(g,z) = f(\pi(g),z)$.

In the case that $f(y,z) = |g|f(z)$, we need to know the differential form, w_φ, at $\pi(e)$. To evaluate w_φ at this point, we shall consider w_G, the differential form on G that generates μ, and w_H, the differential form on $H = G_Z$ that generates μ_H, the Haar measure on H such that $\mu_H(H) = 1$. By a slight generalization of Helgason's [8], theorem 1.7 on page 369, if k is integrable on G,

$$(11) \qquad \int_G k(g)\mu(dg) = \int_Y \int_H k(gh)\mu_H(dh)\mu_Y(dy)$$

where the measures are suitably normalized since $\mu_Y = \mu\pi^{-1}$ and $\mu_H(H) = 1$. In a small enough neighborhood V of $e \in G$ where $(Y \cap V) \times (H \cap V)$ is analytically homeomorphic to V (Chevalley [3], page 109-110), (11) implies that $\mu = \mu_Y\mu_H$ and so, by possibly taking a smaller neighborhood of e, $w_G = \pm\, w_\varphi \wedge w_H$. By means of the relation, $w_G = \pm\, w_\varphi \wedge w_H$, we shall be able to evaluate w_φ at $\pi(e)$.

CHAPTER VI
NOTATION FOR APPLICATIONS

In the following applications the element of the matrix C in the i-th row and j-th column will be denoted c_{ij}, i.e., $C = \{c_{ij}\}$. If the Lie group G can be represented by the collection of matrices, $\{C\}$, c_{ij} is an analytic function on G and its differential is written dc_{ij}. We let $dC = \{dc_{ij}\}$. If the elements of any matrix, M, are analytic functions on an analytic manifold, we write $dM = \{dm_{ij}\}$. If A is a matrix of constants and B and C are matrices of functions, we have $d(AB) = AdB$, $d(BC) = (dB)C + BdC$ and $d(B + C) = dB + dC$.

Since we are dealing with measures we will assume that all Jacobians are positive, that is, we shall always take the absolute value. Furthermore, $|C|$ will denote the absolute value of the determinant of the matrix C and we shall write $\prod_{C} dc_{ij} = dc_{11} \wedge \dots \wedge dc_{mn}$, where C is $m \times n$ and we can ignore the possible change of sign caused by reordering the dc_{ij}. The measure, $\mu(dv)$, on the manifold, V, generated by the differential form, $f(v)dv_1 \wedge \dots \wedge dv_n$, will be written, $\mu(dv) = f(v) \prod_{i=1}^{n} dv_i$.

The notation $\prod_{C} c_{ij}$ shall denote either the product of all elements of the matrix C or the product of all elements that are part of a coordinate system on the manifold of which the matrix is an element; the case being clear from the context. If the subscripts in the above notation are written with a relation between them, one takes the product of all elements satisfying the relation, e.g., $\prod_{T} t_{ii}$ is the product of all elements t_{ij} in the matrix T such that $i = j$, i.e., if T is square, the diagonal elements.

As usual if C is a matrix, C' is its transpose, $\mathrm{tr}C$ its trace, $\mathrm{etr}C$ the exponential of its trace and, if C is nonsingular, C^{-1} is its inverse. I_p denotes the $p \times p$ identity matrix. If the order is clear, we merely write I. A matrix T is upper (lower) triangular if all elements below (above) the principal diagonal are zero; strictly upper (lower) triangular means that the elements on the principal diagonal are also zero. Orthogonal matrices will usually be written as Ω with or without sub- or superscripts, while matrices all of whose elements are zero will be written O. Positive definite matrices will always be symmetric and, if a matrix, S, is symmetric, $S > 0$ means it is positive definite. If a matrix, C, is diagonal in blocks, i.e., $C = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_n \end{pmatrix}$ where the C_i are square matrices, we shall write $C = \mathrm{diag}(C_1, C_2, \ldots, C_n)$. (The blocks may consist of single elements.) $GL(n,R)$ shall denote the general linear group of order n, i.e., the group of all real $n \times n$ nonsingular matrices. The group of all $n \times n$ orthogonal matrices will be written $O(n)$.

CHAPTER VII
EXPLICIT FORMULAS FOR HAAR MEASURES

We shall need to know explicit expressions for various Haar measures. If a matrix Lie group is an open submanifold of E^p, then the calculation of a Haar measure on G is usually quite easy since any such Haar measure and Lebesgue measure are absolutely continuous with respect to each other. In fact, if Lebesgue measure restricted to G is $\mu_p(dg)$ and a Haar measure on G is $\mu_G(dg)$, then $\mu_G(dg) = f(g)\mu_p(dg)$ where f is a positive analytic function. We fix $f(e) = 1$, e being the identity, as any μ_G will suffice for our needs. Now $\mu_G(g_1 dg) = f(g_1 g)\mu_p(g_1 dg) = f(g_1 g)|g_1|\mu_p(dg)$ where $|g_1|$ is the absolute value of the Jacobian of the linear transformation $g_1\colon G \longrightarrow G$. By the G-invariance, $\mu_G(g_1 dg) = \mu_G(dg)$ and so $f(g_1 g)|g_1| = f(g)$. Letting $g_1 = g$ and $g = e$, we obtain $f(g) = |g|^{-1}$. Thus $\mu_G(dg) = |g|^{-1}\mu_p(dg)$.

If $G = GL(n,R)$, the usual chart consists of all n^2 elements of the matrices. If $C \in G$, the map $C\colon C_1 \longrightarrow CC_1$ can be considered as the linear transformation of each column vector of C_1 by the matrix C. Since there are n columns, the Jacobian is $|C|^n$ and so $\mu_G(dg) = |C|^{-n} \prod_C dc_{ij}$.

If G consists of the $(p+q) \times (p+q)$ nonsingular matrices of the form, $C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}$, where C_{11} is $p \times p$ and C_{22} is $q \times q$, it is easily seen that in the transformation, $C\colon C_1 \longrightarrow CC_1$, $C, C_1 \in G$, the whole matrix C operates on the first p columns of C_1, while only the submatrix C_{22} operates on the lower right hand block of C_1 which consists of q columns. Thus the Jacobian is $|C|^p |C_{22}|^q = |C_{11}|^p |C_{22}|^{p+q}$ and so $\mu_G(dg) = |C_{11}|^{-p} |C_{22}|^{-p-q} \prod_C dc_{ij}$.

If G consists of the matrices of some fixed order that are lower diagonal in blocks with the i-th block, C_i, being of order q_i, an induction argument immediately gives the Jacobian as $\prod |C_i|^{p_i}$, $p_i = \sum_{j=1}^{i} q_j$, and so $\mu_G(dg) = \prod |C_i|^{-p_i} \prod_C dc_{rs}$. If G consists of the lower triangular matrices of order $n \times n$, the above result specializes to $\mu_G(dg) = \prod |c_{ii}|^{-i} \prod_C dc_{ij}$, the absolute value sign being dropped if one considers the subgroup with positive diagonal elements. Should the matrices be upper triangular,

$\mu_G(dg) = \prod |c_{ii}|^{n-i} \prod_{C} dc_{ij}$ is the left invariant measure.

By taking the transpose of the lower triangular matrices, one sees that the right invariant measure on the group of upper triangular matrices is $\mu_G(dg) = \prod_{C} |c_{ii}|^{-i} \prod_{C} dc_{ij}$.

CHAPTER VIII
THE DISTRIBUTION OF THE ROOTS OF $|A - \lambda B| = 0$ WITH A AND B BEING POSITIVE DEFINITE

Let X be the collection of the ordered pairs of $p \times p$ positive definite matrices (A,B) such that the eigenvalues of AB^{-1} are distinct. The group G acting on X is the general linear group, $GL(p,R)$, with the action being $C: (A,B) \longrightarrow (CAC', CBC')$, $C \in G$. Efforts to find a matrix group H that acts as a Lie transformation group on X and satisfies the hypotheses of either of Wijsman's [17] theorems proved fruitless. However a candidate for the role of the submanifold Z in theorem 2 is the set of all (D,I_p) where $D = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\lambda_1 > \lambda_2 > \ldots \lambda_p > 0$. We will show that G, X and the above submanifold, which we shall call Z, satisfy the hypothesis of theorem 2.

If $C \in G_Z$, then $(CDC', CC') = (D,I)$ and so C is orthogonal and $CDC' = D$. Since $C = DCD^{-1}$, $c_{ij} = \lambda_i c_{ij} \lambda_j^{-1}$, and since $\lambda_i \neq \lambda_j$ unless $i = j$, $c_{ij} = 0$, if $i \neq j$. Thus C is diagonal with the diagonal elements being $+1$'s and/or -1's. If C has this form then clearly $C \in G_Z$. G_Z is well defined as the above argument holds for any point of Z. To see that $gz_1 = z_2$ implies that $g \in G_Z$, let $C \in G$, $(D_1,I) \in Z$, $(D_2,I) \in Z$ and $(CD_1C', CC') = (D_2,I)$.

Thus C is orthogonal and D_1 and $CD_1C' = D_2$ have the same eigenvalues. As the definition of Z orders the eigenvalues, $D_1 = D_2$ and so $C \in G_Z$. To check that $GZ = X$, let $(A,B) \in X$ and $C = T\Omega$, where Ω is orthogonal and $TT' = B$. Fix Ω so that $\Omega' T^{-1} A T'^{-1} \Omega = D$, which is possible since the eigenvalues of $T^{-1}AT'^{-1}$ and AB^{-1} are the same. Then $(CDC', CC') = (A,B)$ and so $X \subset GZ$. That $GZ \subset X$ is clear and so the hypotheses of the theorem are satisfied.

We now wish to construct the invariant measure on Y. Since G_Z is a finite group, it is especially easy as Y may be considered in a neighborhood of each point of Y as a submanifold of G having the same dimension as G. Each point of Y is the image of 2^p points of G. Thus the restriction of the Haar measure of G to Y multiplied by 2^p is the induced G-invariant measure on Y. The chart that we shall use in a neighborhood of $\pi(e) \in Y$ is the set of elements of the matrices of G whose first nonzero element in each column is positive. This choice can be made because of the structure of G_Z. In terms of this same chart considered on G, Haar measure has the form $\mu(dg) = |C|^{-p} \prod_C dc_{ij}$. Therefore by the above argument, $\mu_Y(dy) =$

$$2^p |C|^{-p} \prod_C dc_{ij}.$$

The chart on all Z is chosen to consist of the diagonal elements of D, $(D,I) \in Z$, while the chart on X consists of the lower left triangular elements of A and B. In terms of these charts, the measure chosen on Z is $\mu_Z(dz) = \prod_{i=1}^{p} d\lambda_i$ and Lebesgue measure on X is

$$\mu_X(dx) = \prod_{i \geq j} da_{ij} db_{ij}.$$

In order to find $f(z)$ where $\mu_X(dx) = f(z)\mu_Y(dy)\mu_Z(dz)$, we shall evaluate the Jacobian, $J(\bar{e},z)$, $\bar{e} = \pi(e)$, for each $z \in Z$. We shall find $J(\bar{e},z)$ from the relation,

$$\prod_{i \geq j} da_{ij} db_{ij} = J(\bar{e},z) \prod_{C} dc_{ij} \prod d\lambda_i.$$

Now $A = CDC'$ and $B = CC'$ and so at $(\bar{e},z)$, $dA = (dC)D + dD + DdC'$, $dB = dC + dC'$. These equations of differentials state that $da_{ii} = 2\lambda_i dc_{ii} + d\lambda_i$, $db_{ii} = 2dc_{ii}$ for $i = 1,\ldots,p$ and $da_{ij} = \lambda_j dc_{ij} + \lambda_i dc_{ji}$, $db_{ij} = dc_{ij} + dc_{ji}$ for $i \leq j < i \leq p$. One can then see that $da_{ii} \wedge db_{ii} = 2dc_{ii} \wedge d\lambda_i$, $i = 1,\ldots,p$ and $db_{ij} \wedge da_{ij} = (\lambda_j - \lambda_i) dc_{ij} \wedge dc_{ji}$, $i \leq j < i \leq p$, and so

$$\prod_{i \geq j} da_{ij} db_{ij} = 2^p \prod_{i > j} (\lambda_j - \lambda_i) \prod_{C} dc_{ij} \prod_{i=1}^{p} d\lambda_i$$

giving $J(\bar{e},z) = 2^p \prod_{i>j} (\lambda_j-\lambda_i)$. Therefore $f(z) = \prod_{i>j} (\lambda_j-\lambda_i)$ and if $p(x)$ is a density on X, the marginal density on Z with respect to $\mu_Z(dz) = \prod_{i=1}^{p} d\lambda_i$ is, by formula (8),

$$\prod_{i>j} (\lambda_j-\lambda_i) \int_G |C|^{2(p+1)} p(gz)|C|^{-p} \prod_C dc_{ij}$$

where $|g| = |C|^{2(p+1)}$ as can be seen in, for example, Deemer and Olkin [5].

The most common situation involving two covariance matrices in multivariate normal analysis arises when $p(x)$ is the product of two Wishart densities with different population covariance matrices. Because of the action of G on the parameter space, we can without loss of generality let one of the parameter covariance matrices be the identity and the other diagonal. So let A have a Wishart (n,p,T) distribution with $T = \text{diag}(\theta_1,\ldots,\theta_p)$, $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_p > 0$, and B, an independent Wishart (m,p,I) distribution. The joint density on (A,B) is $p(A,B) =$

$$k(p,n,m)|T|^{-(n/2)}|A|^{-\frac{1}{2}(n-p-1)}|B|^{-\frac{1}{2}(m-p-1)} \text{etr}-\tfrac{1}{2}(AT^{-1}+B),$$

where $k(p,n,m) = 1/2^{\frac{1}{2}(n+m)p}\pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p} [\Gamma(\frac{1}{2}(n-i+1))\Gamma(\frac{1}{2}(m-i+1))]$.

As a function on GZ, $p(gz) =$

$$k(p,n,m)|T|^{-\frac{1}{2}n}|C|^{m+n-2(p+1)}|D|^{\frac{1}{2}(n-p-1)}\operatorname{etr}-\tfrac{1}{2}(CDC'T^{-1} + CC').$$

Not writing the constant, $k(p,n,m)$, the marginal density, i.e., the density of D, with respect to $\prod_{i=1}^{p} d\lambda_i$ is

$$(12)\quad \Big(\prod_{i=1}^{p}\lambda_i\Big)^{\frac{1}{2}(n-p-1)}\prod_{i>j}(\lambda_j-\lambda_i)\Big(\prod_{i=1}^{p}\theta_i\Big)^{-\frac{1}{2}n}\times$$

$$\int_G |C|^{m+n}\operatorname{etr}-\tfrac{1}{2}(CDC'T^{-1} + CC')|C|^{-p}\prod_C dc_{ij}.$$

In the special case that $T = \sigma^2 I$, the density of D, (12), becomes

$$(13)\quad \sigma^{-np}\Big(\prod_{i=1}^{p}\lambda_i\Big)^{\frac{1}{2}(n-p-1)}\prod_{i>j}(\lambda_j-\lambda_i)\times$$

$$\int_G |C|^{m+n}\operatorname{etr}-\tfrac{1}{2}(C(\sigma^{-2}D+I)C')|C|^{-p}\prod_C dc_{ij}.$$

Making the transformation, $C \longrightarrow C(\sigma^{-2}D+I)^{\frac{1}{2}} = R$, the integral becomes

$$|I+\sigma^{-2}D|^{-\frac{1}{2}(m+n)}\int_G |R|^{m+n}\operatorname{etr}-\tfrac{1}{2}RR'\,|R|^{-p}\prod_R dr_{ij} =$$

$$|I+\sigma^{-2}D|^{-\frac{1}{2}(m+n)}\ \pi^{\frac{1}{2}p^2}\ 2^{\frac{1}{2}(m+n)p}\prod_{i=1}^{p}\left(\Gamma(\tfrac{1}{2}(m+n-i+1)/\Gamma(\tfrac{1}{2}(p-i+1))\right)$$

and so the density (13) becomes

$$\prod_{i=1}^{p}\frac{\Gamma(\frac{1}{2}(m+n-i+1))\lambda_i^{\frac{1}{2}(n-p-1)}(1+\lambda_i/\sigma^2)^{-\frac{1}{2}(m+n)}}{\Gamma(\frac{1}{2}(n-i+1))\Gamma(\frac{1}{2}(m-i+1))\Gamma(\frac{1}{2}(p-i+1))}\ \times$$

$$\pi^{\frac{1}{2}p}\sigma^{-np}\prod_{i>j}(\lambda_j-\lambda_i)$$

which is to be found with $\sigma^2 = 1$ in Anderson [1], page 315. The distribution in the general case (12) has been found by Constantine in terms of hypergeometric functions of a matrix argument and is listed in James [10], page 484.

CHAPTER IX
CANONICAL CORRELATION COEFFICIENTS

To calculate the distribution of the set of canonical correlation coefficients, we shall consider our space X to consist of the $(p+q) \times (p+q)$ positive definite matrices S, $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, S_{11} being $p \times p$, S_{22} being $q \times q$, $p \leq q$ and the rank of $S_{12} = p$. In the central case one can reduce the problem to that of determining the distribution of the roots of a determinantal equation as is done in Anderson [1], page 323. However, if the parameter correlation coefficients are not zero, the lack of independence makes the approach we use easier for our method. The invariance group G consists of the $(p+q)^2$ matrices, $C = \operatorname{diag}(C_1, C_2)$, C_1 being p^2 and C_2 being q^2. The action of G on X is $S \longrightarrow CSC'$.

Attempts to find a Lie group H so that either of Wijsman's theorems could be applied were unsuccessful. However, a candidate for a global cross section is easily found. This submanifold, called Z as usual, is the set of matrices,

$$D = \begin{pmatrix} I_p & L \\ L' & I_q \end{pmatrix}, \quad L = (\Lambda, 0), \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p),$$

$\lambda_1 > \ldots > \lambda_p > 0$. The λ_i are the square roots of the canonical correlation coefficients. If $p = 1$, λ_1^2 is the multiple correlation coefficient.

We will now show that the hypotheses of the analytic homeomorphism theorem, theorem 2, are satisfied. It is useful to partition C_2 into $\begin{pmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{pmatrix}$, C_{22} being p^2 and C_{33} being $(q-p)^2$. We shall assume that $p < q$. If not, the modification is obvious.

If $C \in G_Z$ (G_Z's existence will be shown) and $D \in Z$, then $D = CDC'$ or $C_1C_1' = I$, $C_1LC_2' = L$ and $C_2C_2' = I$. Thus C_1 and C_2 are orthogonal and $C_1 \Lambda C_{22}' = \Lambda$, $C_1 \Lambda C_{32}' = 0$. Since C_1 and Λ are non-singular, $C_{32}' = 0$ and so $C_{23} = 0$ as C_2 is orthogonal. Thus C_{22} is orthogonal with $C_1 \Lambda C_{22}' = \Lambda$ and therefore $C_{22} = \Lambda^{-1}C_1\Lambda = (C_{22}')^{-1} = \Lambda C_1 \Lambda^{-1}$. The equality of the second and fourth members of the equations above says that $c_{ij}\lambda_j/\lambda_i = c_{ij}\lambda_i/\lambda_j$, $C = (c_{ij})$, and, since $\lambda_i \neq \lambda_j$ unless $i = j$, $c_{ij} = 0$. So C is diagonal with plus and/or minus 1's on the diagonal. Thus C_1 and Λ commute, giving $C_{22} = \Lambda^{-1}C_1\Lambda = C_1$. Therefore, if $C \in G_Z$, which is well defined as D was arbitrary, $C = \mathrm{diag}(C_1, C_{22}, C_{33})$, $C_1 = C_{22}$, $C_1 = \mathrm{diag}(+1\text{'s and }-1\text{'s})$ and $C_{33}C_{33}' = I$. G_Z is compact as it is 2^p copies of the orthogonal group $O(q-p)$.

To verify the second condition, we must show that if $CD_1C' = D_2$, $D_1D_2 \in Z$, then $C \in G_Z$. As in the previous paragraph, C_1 and C_2 are orthogonal and so we consider the equations, $C_1\Lambda_1C_{22}' = \Lambda_2$ and $C_1\Lambda_1C_{32}' = 0$, where Λ_i is the Λ block of D_i, $i = 1,2$. Again $C_{32} = 0$ implying that $C_{23} = 0$ and C_{22} and C_{33} are orthogonal. Since $C_{22} = \Lambda_2^{-1}C_1\Lambda_1 = \Lambda_2C_1\Lambda_1^{-1}$ because $C_{22} = (C_{22}')^{-1}$, we see that $c_{ij}\lambda_j/\theta_i = c_{ij}\theta_i/\lambda_j$, $C_1 = (c_{ij})$, $\Lambda_1 = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$, $\Lambda_2 = \mathrm{diag}(\theta_1,\ldots,\theta_p)$ and so $\lambda_j^2c_{ij} = \theta_i^2c_{ij}$. For fixed i there is one and only one j such that $c_{ij} \neq 0$. There is one since C_1 is nonsingular. If for some i, there are two, say j_1 and j_2, then $\lambda_{j_1} = \lambda_{j_2}$, a contradiction. Thus for each i, there is a unique j such that $\theta_i = \lambda_j$. No two i's have the same j's associated with them, since this fact would imply that the two θ_i's are equal. Since $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$ and $\theta_1 > \theta_2 > \ldots > \theta_p > 0$, we have $\theta_i = \lambda_i$ and $D_1 = D_2$ implying that $C \in G_Z$. Condition (3) has been shown by Hotelling [9].

The elements of $\overline{G} = Y$ can be chosen to have the form, $B = \mathrm{diag}(B_1,B_2)$, $B_2 = \begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix}$, where B_1 is p^2; B_2 is q^2; B_{22} is p^2 and B_{33} is lower triangular with positive diagonal elements, since each element $C \in G$ can be written in the form, $C = BU$, where B is of the form above and $U \in G_Z$. Because of the +1's and -1's on the diagonal of U, we shall also require that the first nonzero element in each column of B_1 be positive. The chart chosen around the image of the identity in Y consists of the elements of the matrices, B, that are not identically zero. We choose as a chart on Z the functions, $\lambda_1, \ldots, \lambda_p$, while the chart on X consists of the lower triangular elements of S. The chart on G consists of the nonidentically zero elements of the matrices, C, i.e., the elements of C_1 and C_2.

Lebesgue measure on X is $\mu_X(dx) = \prod_{i \geq j} ds_{ij}$. We take the measure on Z to be $\mu_Z(dz) = \prod_{i=1}^{p} d\lambda_i$ and, at $\pi(e)$, the measure on Y is $\mu_Y(dy) = k \prod_B db_{ij}$. The Haar measure on G is $\mu(dg) = |C_1|^{-p}|C_2|^{-q} \prod_C dc_{ij}$, while the Haar measure at $e \in H = G_Z$ is $\mu_H(dh) = 2^{-p}A(q-p) \prod_{i<j} dw_{ij}$, $\mu_H(H) = 1$, where the chart on $H = G_Z$ consists of the strictly upper triangular elements of the orthogonal, $(q-p)^2$, block C_{33} of the elements of G_Z. The 2^{-p} arises since G_Z is a collection of 2^p copies of $O(q-p)$.

$A(q-p) = 2^{p-q} \prod_{i=1}^{q-p} \Gamma(i/2)/\pi^{\frac{1}{4}(q-p)(q-p+1)}$ (James [10]).

We shall first determine k in $\mu_Y(dy) = k \prod_B db_{ij}$.
In a neighborhood of $e \in G$, $C = BU$, $C \in G$, $B \in Y$ and $U \in G_Z$, $U = \mathrm{diag}(R,R,\Omega)$, $R = \mathrm{diag}(+1's$ and/or $-1's)$ is p^2 and Ω is $(q-p)^2$ orthogonal. At e, $dC = dB + dU$, which implies that $\prod_C dc_{ij} = \prod_B db_{ij} \prod_{i<j} dw_{ij}$. Thus at e, $\mu_Y(dy) = (2^p/A(q-p)) \prod_B db_{ij}$, i.e., $k = 2^p/A(q-p)$.

We now shall determine $f(z)$ where at $(\pi(e),z)$, $\mu_X(dx) = f(z)\, \mu_Y(dy)\, \mu_Z(dz)$. This shall be done by finding $J(z)$, where $\prod_{i \geq j} ds_{ij} = J(z) \prod_B db_{ij} \prod_{i=1}^{p} d\lambda_i$ at $(\pi(e),z)$.

In terms of the charts mentioned above the analytic homeomorphism between X and $Y \times Z$, near $Z \subset X$, has the form, $S = BDB'$ and so $dS = (dB)D + dD + DdB'$ at $B = I$. This matrix equation of differentials can be partitioned into:

$$dS_{11} = dB_1 + dB_1' \qquad\qquad dS_{22} = dB_{22} + dB_{22}'$$
$$dS_{21} = (dB_{22})\Lambda + d\Lambda + \Lambda dB_1' \qquad\qquad dS_{32} = dB_{32} + dB_{23}'$$
$$dS_{31} = (dB_{32})\Lambda \qquad\qquad dS_{33} = dB_{33} + dB_{33}' .$$

From these equations we find that:

$$\prod_{i=1}^{p} ds_{i+p,i} \prod_{j=1}^{p+q} ds_{jj} = 2^{p+q} \prod_{i=1}^{p+q} db_{ii} \prod_{j=1}^{p} d\lambda_j ;$$

for $0 < j < i \leq p$, we have

$$ds_{ij} \wedge ds_{i+p,j} \wedge ds_{j+p,i} \wedge ds_{i+p,j+p}$$

$$= (db_{ij} + db_{ji}) \wedge (\lambda_j db_{i+p,j+p} + \lambda_i db_{ji})$$

$$\wedge (\lambda_i db_{j+p,i+p} + \lambda_j db_{ij}) \wedge (db_{i+p,j+p} + db_{j+p,i+p})$$

$$= (\lambda_j^2 - \lambda_i^2) db_{ij} \wedge db_{ji} \wedge db_{i+p,j+p} \wedge db_{j+p,i+p}$$

and for $i = 1,\ldots,q-p$, $j = 1,\ldots,p$, we get

$$ds_{i+2p,j} \wedge ds_{i+2p,j+p} = \lambda_j db_{i+2p,j+p} \wedge db_{j+p,i+2p}.$$

As a result,

$$\prod_{i \geq j} ds_{ij} = 2^{p+q} \prod_{i>j} (\lambda_j^2 - \lambda_i^2) \prod_{j=1}^{p} \lambda_j^{q-p} \prod_B db_{ij} \prod_{i=1}^{p} d\lambda_i,$$

and so $J(z) = 2^{p+q} \prod_{i>j} (\lambda_j^2 - \lambda_i^2) \prod_{j=1}^{p} \lambda_j^{q-p}$ and

$$f(z) = 2^q A(q-p) \prod_{i>j} (\lambda_j^2 - \lambda_i^2) \prod_{j=1}^{p} \lambda_j^{q-p}.$$

Since the Jacobian of the action of $C \in G$ on X is $|C|^{p+q+1}$, the marginal density on Z of a density, $p(x)$, on X is, by equation (8),

$$(14) \qquad 2^q A(q-p) \prod_{i>j} (\lambda_j^2 - \lambda_i^2) \prod_{j=1}^{p} \lambda_j^{q-p} \int_G |C|^{p+q+1} p(DCD')\mu(dg)$$

with respect to $\mu_Z(dz) = \prod_{i=1}^{p} d\lambda_i$.

In the usual multinormal case, the density $p(x)$ is the Wishart $(m,p+q,\Sigma)$ density. Due to the action of G on the parameter space, we obtain the same marginal density on Z for every point on a G-orbit in the parameter space. Since the action of G is $\Sigma \longrightarrow C\Sigma C'$, the action of G on Σ^{-1} is $\Sigma^{-1} \longrightarrow (C\Sigma C')^{-1} = C'^{-1}\Sigma^{-1}C^{-1}$. However $C'^{-1} \in G$ so that G acts on the inverse of parameter covariance matrices as it does on the matrices themselves and so we can always choose a point on each parameter space orbit to be of the form $\Sigma^{-1} = \begin{bmatrix} I_p & M \\ M' & I_q \end{bmatrix}$, M: $p \times q$, $M = (\Delta, 0)$, Δ: p^2, $\Delta = \text{diag}(\delta_1, \ldots, \delta_p)$, $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_p \geq 0$. The density on X has the form

$$p(S) = 2^{-\frac{1}{2}n(p+q)} \Gamma_{p+q}^{-1}(n/2)\ |\Sigma|^{-\frac{1}{2}n} |S|^{\frac{1}{2}(n-p-q-1)} \text{etr}\ -\tfrac{1}{2}S\Sigma^{-1}$$

where $\Gamma_p(n/2) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(\frac{1}{2}(n-i+1))$. As $|\Sigma|^{-\frac{1}{2}n} = \prod_{i=1}^{p} (1-\delta_i^2)^{-\frac{1}{2}n}$ and $|D|^{\frac{1}{2}(n-p-q-1)} = \prod_{i=1}^{p} (1 - \lambda_i^2)^{\frac{1}{2}(n-p-q-1)}$,

$$p(CDC') = 2^{-\frac{1}{2}n(p+q)}\Gamma_{p+q}^{-1}(n/2)\prod_{i=1}^{p}[(1-\delta_i^2)^{-\frac{1}{2}n}(1-\lambda_i^2)^{\frac{1}{2}(n-p-q-1)}]$$

$$\times |C|^{n-p-q-1} \operatorname{etr} -\tfrac{1}{2}CDC'\Sigma^{-1}$$

and the density of the λ_i with respect to $\prod_{i=1}^{p} d\lambda_i$ is

$$(15) \qquad 2^q A(q-p) 2^{-\frac{1}{2}n(p+q)}\Gamma_{p+q}^{-1}(n/2)$$

$$\times \prod_{i=1}^{p}[(1-\delta_i^2)^{-\frac{1}{2}n}(1-\lambda_i^2)^{\frac{1}{2}(n-p-q-1)}]$$

$$\times \prod_{i>j}(\lambda_j^2-\lambda_i^2)\prod_{j=1}^{p}\lambda_j^{q-p}$$

$$\times \int_G |C|^n \operatorname{etr} -\tfrac{1}{2}CDC\,\Sigma^{-1}|C_1|^{-p}|C_2|^{-q}\prod_C dc_{ij}.$$

In the case that the $\delta_i = 0$, $\Sigma^{-1} = I$ and $trCDC' = trC_1C_1' + trC_2C_2'$, the integral in the density then becomes

$$\int_{C_1} |C_1|^n \operatorname{etr} -\tfrac{1}{2}C_1C_1' \; |C_1|^{-p}\prod_{C_1} dc_{ij}$$

$$\times \int_{C_2} |C_2|^n \operatorname{etr} -\tfrac{1}{2}C_2C_2' \; |C_2|^{-q}\prod_{C_2} dc_{ij}$$

which is equal to

$$\pi^{\frac{1}{2}(p^2+q^2)}\, 2^{\frac{1}{2}n(p+q)} \prod_{i=1}^{p} \frac{\Gamma(\frac{1}{2}(n-i+1))}{\Gamma(\frac{1}{2}(p-i+1))} \prod_{i=1}^{q} \frac{\Gamma(\frac{1}{2}(n-i+1))}{\Gamma(\frac{1}{2}(q-i+1))}$$

and so the density in this central case is

$$(16) \qquad 2^p \pi^{\frac{1}{2}p} \prod_{i=1}^{p} \frac{\Gamma(\frac{1}{2}(n-i+1))}{\Gamma(\frac{1}{2}(n-q-i+1))\Gamma(\frac{1}{2}(p-i+1))\Gamma(\frac{1}{2}(q-i+1))}$$

$$\times \prod_{i=1}^{p} (1 - \lambda_i^2)^{\frac{1}{2}(n-p-q+1)} \prod_{i>j} (\lambda_j^2 - \lambda_i^2) \prod_{j=1}^{p} \lambda_j^{q-p}$$

with respect to $\prod_{i=1}^{p} d\lambda_i$. This result (16) can be checked with Anderson [1], page 324, equation (13) by letting $p = p_1$, $q = p_2$ and $f_i = \lambda_i^2$.

CHAPTER X
MULTIVARIATE F AND HOTELLING'S T^2

The following situation arises when one has k p-dimensional multinormal populations with distributions $N(u_i,\Sigma)$ and one wishes to test the hypothesis $u_i = 0$, $i = 1,\ldots,k$ versus at least one of the u_i is not a zero vector. If $k = 1$, the problem is one of deciding whether or not a population has a zero mean. Wijsman's [17] theorem 6 applies so that the problem does not use the results of this thesis. However, the density found will be of use in recognizing the form of a later result.

Let $X = \{(U,S) | U$, a $p \times k$ matrix, $k \leq p$; S, a p^2 positive definite matrix; $U'S^{-1}U > 0\}$. In applications U is the set of sample means and S is a sample covariance matrix. X is open in E^r, $r = pk + \frac{1}{2}p(p+1)$. The invariance group, G, consists of all p^2 nonsingular matrices, i.e., $GL(p,R)$, and, if $C \in G$, the action of C on X is $(U,S) \longrightarrow (CU,CSC')$. It is useful to partition C into $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, where C_{11} is k^2 and C_{22} is $(p-k)^2$ and to partition U into $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ where U_1 is k^2. We shall show that Wijsman's [17] theorem 6 applies with $x_o = (\delta, I_p)$, $\delta' = (I_k, 0)$, and H being the group of all k^2 upper triangular matrices, T, with positive diagonal elements. The action of H on X is $(U,S) \longrightarrow (UT,S)$. That G and H commute is clear.

G_{x_o} is compact, since $(C\delta, CC') = (\delta, I)$ implies that C is orthogonal. Since $C\delta = \delta$, $C_{11}I = I$ and $C_{21}I = 0$. With the orthogonality of C, these facts imply that if $C \in G_{x_o}$, $C = \mathrm{diag}(I_k, \Omega)$, $\Omega \in O(p-k)$. H_{x_o} is the trivial subgroup of H since $(\delta T, I) = (\delta, I)$ implies that $T = I$ and so condition (i) of Wijsman's theorem 6 is satisfied.

If $(C\delta, CC') = (\delta T, I)$, then C is orthogonal, $CC' = I$, and $C\delta = \delta T$ from which it is seen that $C \in G_{x_o}$, $T \in H_{x_o}$ and condition (ii) is satisfied. (iii) is clear. To check condition (iv), if $(U,S) \in X$, let C_1 be lower triangular such that $CC_1' = S$. Then $(C_1^{-1}U, I) \in X$, since $(C_1^{-1}U)'C_1^{-1}U = U'C_1'^{-1}C_1{}^{-1}U = U'S^{-1}U > 0$. Since $C_1{}^{-1}U$ has rank k, there exists an orthogonal matrix, Ω, such that $\Omega' C_1{}^{-1}U = \begin{pmatrix} T \\ 0 \end{pmatrix} = \delta^*$ where $T \in H$. Letting $C = C_1\Omega$, then $(C\Omega\delta^* T, CC') = (U,S)$ and so $X \subset GZ$. That $GZ \subset X$ is trivial and so Wijsman's theorem 6 applies.

As $Z = Hx_o$, we have by Wijsman [17], theorem 6, the induced density on Z from a density, $p(x)$, on X as

$$(17) \qquad c|h| \int_G p(ghx_o)\,|g|\,\mu(dg)$$

with respect to $\mu_Z(dz)$, where

$$(18) \qquad c = \mu_X(dx)/\mu_Y(dy)\mu_Z(dz)$$

and μ_X is Lebesgue measure on X, μ_Y is the invariant measure on $Y = G/G_{x_o}$ induced by the Haar measure, μ, on G, $|h|$ is the absolute value of the Jacobian of the linear transformation, $h: X \longrightarrow X$ and similarly with $|g|$.

In order to calculate c in equation (18), it is necessary to choose charts on G, Y, G_{x_o} and Z. The chart chosen on G consists of the elements of the matrices $C \in G$. Since every element of G can be written in the form, $C = B\theta$, where $\theta \in G_{x_o}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, B_{11} being k^2, B_{22} being $(p-k)^2$ and B_{22} being lower triangular with positive diagonal elements, we can identify Y with the collection of matrices, B, and let the nonidentically zero elements of B be a chart on Y. A chart on G_{x_o} near the identity consists of the strictly upper triangular elements of the lower right hand $(p-k)^2$ block, Ω, where $\theta = \operatorname{diag}(I_k,\Omega)$ and $\Omega = (w_{ij})$. First it is necessary to evaluate k, where $\mu_Y(dy) = k \prod_B db_{ij}$ at $\pi(e)$, the image of the identity in Y. Since $C = B\theta$ near the identity, $dC = dB + d\theta = dB + \mathrm{diag}(0, d\Omega)$ at the identity and so $\prod_C dc_{ij} = \prod_B db_{rs} \prod_{l<m} dw_{lm}$. Thus $\mu_Y(dy) = (1/A(p-k)) \prod_B db_{ij}$ where $A(p-k)$ is the normalizing constant at the identity so that the invariant measure on $G_{x_o} = O(p-k)$ assigns measure one to G_{x_o}. $A(p-k) = 2^{k-p} \prod_{i=1}^{p-k} \Gamma(i/2)/\pi^{\frac{1}{4}(p-k)(p-k+1)}$ (James [10]). To determine c in (18), we note that the analytic homeomorphism near x_o is expressed by the equations, $U = B\delta T$ and $S = BB'$, so that at x_o, $dU = (dB)\delta + \delta dT$ and $dS = dB + dB'$ implying that

$$\mu_X(dx) = \prod_U du_{ij} \prod_{r\geq m} ds_{rm} = 2^p \prod_B db_{ij} \prod_T dt_{ij} \tag{19}$$

at x_o, where the elements of T are a chart on $Hx_o = Z = H$. Combining (18) and (19), we find that $c = 2^p A(p-k)$.

In terms of the given charts, $\mu(dg) = |C|^{-p} \prod_C dc_{ij}$, $\mu_H(dh) = \mu_Z(dz) = \prod_{i=1}^{k} (t_{ii})^{-i} \prod_T dt_{ij}$ and $|h| = |T|^p$. Also

$|g| = |C|^{k+p+1}$ using Deemer and Olkin [5]. Thus the marginal density (17) with respect to μ_Z on Z is

$$2^p A(p-k)|T|^p \int_G p(ghx_o)|C|^{k+1} \prod_C dc_{ij}.$$

In the usual multinormal situation, the columns of U are independent normal vectors with $E(U) = \xi$ and common covariance matrix, Σ, and S has a Wishart (n,p,Σ) distribution independent of U. From the action of G on the parameter space, we obtain the same marginal density if we replace Σ with $\Sigma^{-\frac{1}{2}}\Sigma\Sigma^{-\frac{1}{2}} = I$ and ξ with $v = \Sigma^{-\frac{1}{2}}\xi$. So we take $p(x)$ to be the product of normal (v,I) and Wishart (n,p,I) densities. The induced marginal density on Z, i.e., that of the matrix T, can be used to find the distribution of $T'T = U'S^{-1}U$.

The explicit form of the density on X is

$$p(U,S) = c(k,n,p)|S|^{\frac{1}{2}(n-p-1)} \operatorname{etr} -\tfrac{1}{2}\{(U-v)(U-v)' + S\} \tag{20}$$

where $c(k,n,p) = 1/(2\pi)^{\frac{1}{2}(k+n)p}\Gamma_p(n/2)$ and

$$\Gamma_p(n/2) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}(n-i+1)).$$

As $U = C\delta T$ and $S = CC'$.

(21) $p(ghx_o) = c(k,n,p)|C|^{n-p-1}\text{etr}-\frac{1}{2}\{(C\delta T-v)(C\delta T-v)' + CC'\}$

and the marginal density with respect to $\mu_Z(dz) = \prod_{i=1}^{k}(t_{ii})^{-i}\prod_T dt_{ij}$ is

(22) $$2^p A(p-k)\prod_{i=1}^{k}(t_{ii})^p\, c(k,n,p) \times$$

$$\int_G |C|^{n-p+k}\text{etr}-\frac{1}{2}\{(C\delta T-v)(C\delta T-v)' + CC'\}\prod_C dc_{ij}.$$

In the case that $k = 1$, then U is a $p \times 1$ normal (v,I) random vector and S is a Wishart (n,p,I) random covariance matrix. It is well-known (Anderson [1], page 106) that $((n-p+1)/p)$ times $T'T = U'S^{-1}U$ (a constant times Hotelling's T^2 statistic) has a noncentral $F_{p,n-p+1}$ distribution with noncentrality parameter, $((n+1)/n)\xi'\Sigma^{-1}\xi$. In the case that $k > 1$, an equivalent distribution, that of $U'S^{-1}U$, has been found by James [10]. James calls it a multivariate F distribution.

We shall refer back to the density (22) in the next chapter.

CHAPTER XI
A GENERALIZATION OF THE MULTIVARIATE F PROBLEM

The following problem was suggested by a paper of Gleser and Olkin in Multivariate Analysis [12], ed. P. R. Krishnaiah. It is a generalization of the result in the previous chapter. This problem arises when one has k $(p+q)$-dimensional multinormal vectors with means u_i, $i = 1,\ldots,k$ and a common covariance matrix, Σ. Partitioning u_i into $\begin{pmatrix} u_i^1 \\ u_i^2 \end{pmatrix}$, one wishes to test the hypothesis that $u_i^1 = 0$, $i = 1,\ldots,k$, versus the hypothesis that at least one of the u_i^1 is not zero. Let $X = \{(U,S) | U$, a $(p+q) \times k$ matrix; $k \leq \min(p,q)$; S, a $(p+q)^2$ positive definite matrix and U and S satisfy condition (*) below$\}$. To state condition (*), we partition S into $\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ where S_{11} is p^2 and S_{22} is q^2, and we partition U into $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ where U_1 is $p \times k$ and U_2 is $q \times k$; then condition (*) is

$$(*) \qquad U_1^{1} S_{11}^{-1} U_1 > 0 \quad \text{and} \quad U' S^{-1} U - U_1' S_{11}^{-1} U_1 > 0.$$

X is seen to be open in E^r, $r = (p+q)k + \frac{1}{2}(p+q)(p+q+1)$. The invariance group, G, consists of the $(p+q)^2$ non-singular matrices of the form, $C = \begin{bmatrix} C_1 & 0 \\ C_2 & C_3 \end{bmatrix}$ where C_1 is p^2 and C_3 is q^2. If $C \in G$, the action of C on X is $(U,S) \longrightarrow (CU, CSC')$. It is useful to partition C_1 into $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, C_{11} being k^2, C_{22} being $(p-k)^2$, and to partition C_3 into $\begin{bmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{bmatrix}$, C_{33} being k^2 and C_{44} being $(q-k)^2$.

In this case neither of Wijsman's [17] theorems were able to be applied. A submanifold, Z, does however exist which satisfies the conditions of our theorem 2. We shall let $Z = \{(\delta, I) | \delta = (\delta_1', \delta_2')' = (T_1', 0', T_2', 0')'\}$, T_1 and T_2 being k^2 upper triangular matrices with positive diagonal elements.

We wish to verify the hypotheses of theorem 2. Let $z = (\delta, I) \in Z$ and $C \in G_z$. If $C\delta = \delta$ and $CC' = I$, then C is orthogonal and so $C_2 = 0$ and C_1 and C_2 are orthogonal. Also $C(T_1', 0')' = (T_1', 0')'$ implies that $C_{11}T_1 = T_1$ and $C_{21}T_1 = 0$ and since T_1 is nonsingular, $C_{11} = I$, $C_{21} = 0$ and $C_{12} = 0$. The same argument applies to C_3 and so if $C \in G_Z$, C has the form, $\mathrm{diag}(I, \Omega_{22}, I, \Omega_{44})$, $\Omega_{22} \in O(p-k)$ and $\Omega_{44} \in O(q-k)$. Clearly if C has this form, $C \in G_Z$. Since G_Z can be considered the direct product of $O(p-k)$ and $O(q-k)$, G_Z is compact.

To see that condition (2) holds, let $z_1 = (\delta^1, I)$ and $z_2 = (\delta^2, I)$ with $\delta^i = (T_1^{i\prime}, 0', T_2^{i\prime}, 0')'$, $i = 1,2$. If $C \in G$ and $(C\delta', CC') = (\delta^2, I)$, then C is orthogonal and so $C_2 = 0$ and C_1 and C_3 are orthogonal. Also $C_{11}T_1^1 = T_1^2$ and $C_{21}T_1^1 = 0$. Since T_1^1 and T_1^2 are non-singular, $C_{21} = 0$, $C_{12} = 0$ and $C_{11} = T_1^2 T_1^{1-1}$. But then C_{11} is upper triangular with positive diagonal elements and also orthogonal, i.e., $C_{11} = I$. The same argument applies to C_3 and so $C \in G_Z$.

To verify condition (3) of theorem 2 we will show that if $(U,X) \in X$, there exists a $C \in G$ such that $(CU, CSC') = (\delta, I)$ for some $(\delta, I) \in Z$ and then that $(C\delta, CC') \in X$ for all $C \in G$. Let C_* be lower triangular such that $C_* C_*' = S$. $C_*^{-1} \in G$ and $C_*^{-1}: (U,S) \longrightarrow (C_*^{-1}U, I)$.

If $C_*^{-1} = \begin{bmatrix} R_1 & 0 \\ R_2 & R_3 \end{bmatrix}$, then $C_*^{-1}U = \begin{bmatrix} R_1U_1 \\ R_2U_1 + R_3U_2 \end{bmatrix}$ and

$U_1'R_1'R_1U_1 = U_1S_{11}^{-1}U_1 > 0$. Also $U'C_*'^{-1}C_*^{-1}U - U_1'R_1'R_1U_1 = U'S^{-1}U - U_1S_{11}^{-1}U > 0$ and so $(C_*^{-1}U, I) \in X$ and we can restrict our attention to elements of X of the form (U,I)

where $U_1'U_1 > 0$ and $U'U - U_1'U_1 = U_2'U_2 > 0$. Since $U_1U_1 > 0$, there exists an orthogonal transformation Ω_1 such that $\Omega_1 U_1 = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}$ where T_1 is k^2, upper triangular with positive diagonal elements. A similar Ω_2 and T_2 exist for U_2. Letting $C = \begin{bmatrix} \Omega_1 & \\ & \Omega_2 \end{bmatrix} \epsilon\ G$, we have $(CU,I) = (\delta,I)$ and so $GZ \supset X$. $GZ \subset X$ since $(C\delta,CC')$ is such that $CC' > 0$ and $(C\delta)_1'(CC')_{11}^{-1}(C\delta)_1 = T_1'T_1 > 0$ and $(C\delta)'(CC')^{-1}C\delta - T_1'T_1 = \delta'\delta - T_1'T_1 = T_2'T_2 > 0$.

So the theorem applies and there exists an open subset of X, X^*, and an open subset of Z, Z^*, such that $G/G_Z \times Z^*$ is analytically homeomorphic to X^*. Actually $X = X^*$ and $Z = Z^*$.

We now turn to the problem of putting charts on our various manifolds. If $C \epsilon G$, there is a unique matrix

$B = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}$ with B_1 being p^2, B_3 being q^2,

$B_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, B_{11} being k^2, B_{22} being $(p-k)^2$

lower triangular with positive diagonal elements,

$B_3 = \begin{bmatrix} B_{33} & B_{34} \\ B_{43} & B_{44} \end{bmatrix}$, B_{33} being k^2, B_{44} being $(q-k)^2$

lower triangular with positive diagonal elements (the same form as C with added restrictions on B_{22} and B_{44}) and a matrix $\Omega \in G_Z$ such that $C = B\Omega$. We can and do let Y be the collection of all such matrices B. The chart chosen on Y consists of all elements of B that are not identically zero. The chart chosen on G is the set of all elements of the matrices, $C \in G$. To see the chart on G_Z near the identity let $\Omega \in G_Z$, $\Omega = \mathrm{diag}(I, \Omega_1, I, \Omega_2)$. The chart chosen on G_Z consists of the strictly upper triangular elements of Ω_1 and Ω_2. We shall let the chart on Z consist of the upper triangular elements of the matrices T_1 and T_2, while the chart on X consists of the elements of U and the lower triangular elements of S.

The first problem is to determine k, where $\mu_Y(dy) = k \prod_B db_{ij}$ at $\pi(e)$, $\mu(dg) = \prod dc_{ij}$ at e and

$\mu_{G_Z}(dz) = A(p-k)A(q-k) \prod_\Omega dw_{ij}$ at e. μ_{G_Z} is chosen so that

$\mu_{G_Z}(G_Z) = 1$ and since G_Z is the direct product of $O(p-k)$ and $O(q-k)$, $A(p-k)$ and $A(q-k)$ are the normalizing constants at the identity of these orthogonal groups. Since $C = B\Omega$ near e, $dC = dB + d\Omega$ at e and, trivially,

$\prod_C dc_{ij} = \prod_B db_{ij} \prod_\Omega dw_{ij}$. Thus $k = 1/(A(p-k)A(q-k))$.

Next we wish to find the relation between Lebesgue measure on X, $\mu_X(dx) = \prod_U du_{ij} \prod_{i\geq j} ds_{ij}$, $\mu_Y(dy)$ and

$\mu_Z(dz)$, which we choose equal to $\prod_{T_1} t_{ii}^{-1} \prod_{T_1} dt_{ij} \prod_{T_2} t_{jj}^{-1} \prod_{T_2} dt_{ij}$,

at the points of $Z \subset X$. This particular choice for a measure on Z is motivated by the similarity between the Z of this problem and the submanifold Z in the previous chapter. The analytic homeomorphism between $Y \times Z$ and X near the points $(\pi(e),z)$ can be expressed by the equations, $U = B\delta$ and $S = BB'$, which imply that at the points of $Z \subset X$, $dU = (dB)\delta + d\delta$ and $dS = dB + dB'$. After straightforward but tedious manipulation, we find that

$$\prod_U du_{ij} \prod_{r\geq m} ds_{rm} = 2^{p+q} \prod_{i=1}^{k} t_{ii}{}^{p-i} \prod_{j=1}^{k} t_{jj}{}^{q-j} \prod_{T_1} dt_{ij} \prod_{T_2} dt_{ij} \prod_B db_{mn}$$

at $(\pi(e),z)$ and so, at $(\pi(e),z)$, using (8)

$$\mu_X(dx) = 2^{p+q}A(p-k)A(q-k) \prod_{i=1}^{k} t_{ii}{}^{p} \prod_{j=1}^{k} t_{jj}{}^{q}\, \mu_Y(dy)\mu_Z(dz)$$

where in both equations $t_{ii} \in T_1$ and $t_{jj} \in T_2$.

As $\mu(dg) = |C_1|^{-p}|C_3|^{-p-q} \prod_C dc_{ij}$, and, since the Jacobian of the map, $S \longrightarrow CSC'$ is $|C|^{p+q+1}$ and the

Jacobian of the map, $U \longrightarrow CU$, is $|C|^k$, $|g| = |C|^{k+p+q+1}$, the marginal density on Z of a density, $p(x)$, on X is

$$(23) \qquad 2^{p+q}A(p-k)A(q-k) \prod_{i=1}^{k} t_{ii}{}^{p} \prod_{j=1}^{k} t_{jj}{}^{q} \quad \times$$

$$\int_G p(gz)\,|C_1|^{k+p+1}\,|C_3|^{k+1} \prod_C dc_{ij}$$

with respect to $\mu_Z(dz) = \prod_{i=1}^{k} t_{ii}{}^{-i} \prod_{j=1}^{k} t_{jj}{}^{-j} \prod_{T_1} dt_{ij} \prod_{T_2} dt_{ij}$.

Comparing (23) with (22) in the previous chapter, one notes a marked similarity. (23) appears to be the product of two densities having the form of (22). The only difficulty is the appearance of the elements of C_2 in $\mu(dg)$, the Haar measure of G, and in $p(gz)$. However, if $p(gz)$ has a sufficiently nice form, as it will in our application, use can be made of the similarity of the two densities.

In the following application we shall ignore any constants multiplying the densities that appear, since these constants will not be needed for the purpose of recognizing the distribution that arises.

The usual situation for this problem has the k columns of U being independent normal vectors with covariance matrix, Σ, $E(U) = \Lambda$ and S being a random matrix with a Wishart $(n,p+q,\Sigma)$ distribution. Because of the action of the invariance group on the parameter space, we take, without loss of generality, the parameter covariance matrix to be the identity and $E(U) = M \doteq \Sigma^{-\frac{1}{2}}\Lambda$ where $M = (M_1', M_2')'$, M_1 being $p \times k$ and M_2 being $q \times k$. So the density on X, ignoring the constant, is

$$(24) \qquad p(U,S) = |S|^{\frac{1}{2}(n-p-q-1)} \operatorname{etr} -\tfrac{1}{2}[(U-M)(U-M)' + S].$$

As $S = CC'$, $|S| = |C|^2$ and $\operatorname{tr}S = \sum_C c_{ij}^2$. Also $U = C\delta$ and thus $\operatorname{tr}(U-M)(U-M)' + \operatorname{tr}S = \operatorname{tr}(C\delta-M_1)(C\delta-M_1)'$
$+ \operatorname{tr}(C_2\delta_1+C_3\delta_2-M_2)(C_2\delta_1+C_3\delta_2-M_2)' + \operatorname{tr}C_1C_1' + \operatorname{tr}C_2C_2' + \operatorname{tr}C_3C_3'$.

Now let us consider the integral factor of (23),

$$\int p(C\delta, CC')|C_1|^{k+q+1}|C_3|^{k+1} \prod_C dc_{ij},$$

and note that C_2 only appears in the exponential term of $p(C\delta, CC')$ and that the last $p-k$ columns only appear in the $\operatorname{etr} -\frac{1}{2}C_2C_2'$ term. Each of the latter terms gives us

$\int \exp -\frac{1}{2}c_{ij}^2 dc_{ij} = (2\pi)^{\frac{1}{2}}$. As the total contribution from these terms is a multiplicative constant, we ignore the constant. Let N denote the first k columns of C_2 and let $K = C_3\delta - M_2$. N only appears in the term, $\operatorname{etr} -\frac{1}{2}[(NT_1+K)(NT_1+K)' + NN']$. If we let $DD' = I + T_1T_1'$, where D is upper triangular with positive diagonal elements, then

$$(25) \qquad (NT_1+K)(NT_1+K)' + NN' = (ND + KT_1'D'^{-1})(ND + T_1'D'^{-1})$$
$$+ K(I - T_1'(DD')^{-1}T_1)K'.$$

As $I - T_1'(DD')^{-1}T_1 = (I + T_1'T_1)^{-1}$ and K contain no elements of N, we shall consider the second term of the above expression (25) later. From the elements of N we have

$$\theta = \int \operatorname{etr} -\tfrac{1}{2}(ND + KT_1'D'^{-1})(ND + KT_1'D'^{-1}) \prod_N dc_{ij} =$$
$$\int \operatorname{etr} -\tfrac{1}{2}NDL'N' \prod_N dc_{ij},$$

since Lebesgue measure is translation invariant. Using the linear transformation, $N \longrightarrow ND = B$, $\prod_N dc_{ij} = |D|^{-q} \prod_B db_{ij}$ since B has q rows and we obtain

$$\theta = \int |D|^{-q} \operatorname{etr} -\tfrac{1}{2}BB' \prod_{B} db_{ij} = |D|^{-q}(2\pi)^{qk}.$$

Again ignoring the constant, we find that the terms of C_2 contribute a factor, $|D|^{-q} = |I+T_1T_1'|^{-\frac{1}{2}q} = |I+T_1'T_1|^{-\frac{1}{2}q}$, the last equality occurring because the eigenvalues of T_1T_1' and $T_1'T_1$ are the same.

Thus (23) takes the form,

$$\begin{aligned}
(26)\quad & \text{const.} \times \prod_{T_1} t_{ii}{}^{p} \int_{C_1} |C_1|^{n-p+k} \\
& \times \operatorname{etr} -\tfrac{1}{2}[(C_1\delta_1+M_1)(C_1\delta_1+M_1)' + C_1C_1'] \prod_{C_1} dc_{ij} \\
& \times |I + T_1T_1|^{-\frac{1}{2}q} \prod_{T_2} t_{jj}{}^{q} \\
& \times \int_{C_3} |C_3|^{n-p-q+k} \operatorname{etr} -\tfrac{1}{2}[(C_3\delta_2-M_2)\psi^{-1}(C_3\delta_2-M_2)' + C_3C_3'] \prod_{C_1} dc_{ij}, \\
& \psi = (I + T_1'T_1),
\end{aligned}$$

with respect to $\mu_Z = \prod_{T_1} t_{ii}^{-i} \prod_{T_1} dt_{ij} \prod_{T_2} t_{jj}^{-j} \prod_{T_2} dt_{ij}$. It is useful to note that $\prod_{T_2} t_{jj}^{-j} \prod_{T_2} dt_{ij}$ is the right invariant

Haar measure on the group of all $k \times k$ upper triangular matrices with positive diagonal elements. If we let $RR' = (I + T_1'T_1)^{-1}$ where R is $k \times k$ upper triangular with positive diagonal elements and use the transformation, $T_2 \longrightarrow T_2R = T_3$, so that $\delta_2 \longrightarrow \delta_2R = \delta_3$, the invariance of the Haar measure makes the second factor of (26),

$$(27)\quad \prod_{T_3} t_{rr}{}^{q} \int_{C_3} |C_3|^{n-p-q+k}$$

$$\times \ \operatorname{etr} -\tfrac{1}{2}[(C_3\delta_3-M_2R)(C_3\delta_3-M_2R)' + C_3C_3'] \prod_{C_3} dc_{ij} .$$

Now let us compare the first factor of (26) and the expression (27) with (22) of the previous chapter remembering that $\delta_1 = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}$ and $\delta_3 = \begin{bmatrix} T_2R \\ 0 \end{bmatrix}$. We have the following correspondence,

(22)	(26)	(27)
C	C_1	C_3
T	T_1	$T_3 = T_2R$
V	M_1	M_2R
k	k	k
p	p	q
n	n	$n-q$.

Thus if $k = 1$, then $T_1'T_1 = U_1'S_{11}^{-1}U_1$ is $(p/n-p+1))$ times a noncentral $F_{p,n-p+1}$ random variable with noncentrality parameter, $((n+1)/n)M_1'M_1$, while from the second factor we see that given $T_1'T_1 = U_1'S_{11}^{-1}U_1$, $T_2'T_2(I+T_1'T_1)^{-1} = (U'S^{-1}U - U_1'S_{11}^{-1}U)/(1 + U_1'S_{11}^{-1}U_1)$ is $(q/(n-p-q+1))$ times a noncentral $F_{q,n-p-q+1}$ random variable with noncentrality parameter $((n-p+1)/(n-p))M_2'M_2\,(1 + U_1'S_{11}^{-1}U_1)$. This result has been shown by D. G. Kabe [11] and an equivalent one by Giri, Kiefer and Stein [6].

Because of the nature of the result, an induction argument immediately gives the following: if U is a $r \times 1$ normal (u,Σ) random vector and S is a Wishart (n,r,Σ) matrix and U_i is the vector consisting of the first p_i components of U, $i = 1,\ldots,m$, $1 \leq p_1 < p_2 < \ldots < p_m = r$, $E(U_i) = u_i$ and S_i is the $p_i{}^2$ upper left block of S, $E(S_i) = \Sigma_i$, Σ_i being the $p_i{}^2$ upper left block of Σ, then $U_1'S_{11}^{-1}U_1$ is $(p_1/(n-p_1+1))$ times a noncentral $F_{p,n-p+1}$ random variable with noncentrality parameter, $((n+1)/n)M_1'M_1$ and $(U_i'S_{ii}^{-1}U_i - U_{i-1}'S_{i-1,i-1}^{-1}U_{i-1})/(1 + U_{i-1}'S_{i-1,i-1}^{-1}U_{i-1})$ is, given $U_{i-1}'S_{i-1,i-1}^{-1}U_{i-1}$, $((p_i-p_{i-1})/(n-p_i+1))$ times a noncentral $F_{p_i-p_{i-1},n-p_i+1}$ random variable with noncentrality parameter, $((n-p_i+1)/(n-p_i+1))M_2'M_2(1 + U_{i-1}'S_{i-1,i-1}^{-1}U_{i-1})$, where $M = (M_1', \ldots, M_m')' = \Sigma^{-1}u$, M_i being $(p_i - p_{i-1}) \times 1$, for all $i = 2, \ldots, m$. The result when G consists of lower triangular matrices is, of course, a special case.

In the case that $k > 1$, we see that $T_1'T_1 = U_1'S_{11}^{-1} U_1$ has a multivariate F distribution with parameters (k,p,n) (James [10] uses (p,m,n)) and noncentrality parameter $M_1'M_1$ and that $R'T_2'T_2R = R'(U'S^{-1}U - U_1'S_{11}^{-1}U_1)R$ has, given $U_1'S_{11}^{-1}U_1$, a multivariate F distribution with parameters $(k,q,n-p)$ and noncentrality parameter, $R'M_2'M_2R$, where R is $k \times k$ upper triangular with positive diagonal elements and $RR' = (I + T_1'T_1)^{-1}$.

APPENDIX
MEASURES GENERATED BY DIFFERENTIAL FORMS

If w_M is a nonzero continuous differential form of maximal degree on the orientable manifold M, i.e., in terms of any chart, say $(m_1,\ldots,m_n)$, on M where $w_M = h(\ )dm_1 \wedge \ldots \wedge dm_n$, h is continuous and nonzero, Chevalley [3] defines an integral of real valued continuous functions with compact support. The support of a function f, $N(f)$, is the closure of the set $\{m \in M | f(m) \neq 0\}$. Chevalley defines a function to have property P if there exists a coordinate neighborhood U with chart $(m_1,\ldots,m_n)$ and a cubic set $V = \{m \in M|\ |m_i(m)| < a \forall i\}$, $V \subset U$, such that f is continuous and $f(x) = 0$ for all $x \in M - V$. For these f one can define $\int f dw_M = \int f^* h^* dm_1 \ldots dm_n$ where f^* is a function on E^n such that $f(m) = f^*(m_1(m),\ldots,m_n(m))$ for each $m \in V$, similarly with h^* and we are evaluating the Riemann integral of f^*h^*. The definition is consistent with any other choice of a coordinate system and coordinate neighborhood.

Given any open set A with compact closure, $Cl(A)$, we can cover $Cl(A)$ and thus A with finitely many cubic sets, V_k, $k = 1,\ldots,p$. Furthermore we can find p

continuous functions, u_k, $k = 1,\dots,p$, such that $1 \geq u_k(m) \geq 0$, $m \in V_k$ and $u_k(m) = 0$, $m \in M - V_k$, and $\sum_{k=1}^{p} u_k(m) = 1$, $m \in A$. (Chevalley [3], page 163, Lemma 1).

If f is continuous and $N(f)$ relatively compact, we define $\int f dw_M = \sum_{k=1}^{p} \int f u_k dw_M$. Chevalley [3] shows that the definition is selfconsistent, i.e., it does not depend on the choice of V_k and u_k.

We denote the class of continuous functions with compact support by F and write $\int f$ for $\int f dw_M$. F is a vector lattice, i.e., if $f,g \in F$ and $a,b \in R$, then $af + bg \in F$, $f \vee g \in F$ and $f \wedge g \in F$ where $\vee$ and $\wedge$ are the standard symbols for maximum and minimum, respectively. The proof of the above statement is simple as $af + bg$, $f \vee g$ and $f \wedge g$ are continuous functions vanishing off the set $N(f) \cup N(g)$. F and its integral, $\int$, form a class of elementary functions with an elementary integral, following the terminology of Asplund and Bungart, A First Course in Integration [2], sections 2.5 and 3.1. That is, F and $\int$ satisfy:

(28) $\int(f+g) = \int f + \int g \qquad f,g \in F$

(29) $\int cf = c\int f \qquad f \in F,\ c \in R$

(30) if $f \geq 0$, then $\int f \geq 0 \qquad f \in F$

(31) if $f_r \geq f_{r+1}$ for each r and $\lim_{r\to\infty} f_r = 0$,

$$\text{then } \lim_{r\to\infty} \int f_r = 0, \quad f_r \in F$$

(32) if $f \in F$, then $1 \wedge f \in F$.

(28), (29) and (30) are obvious from the definition of the integral and (32) from that of F. To show (31), we note that $N(f_1) \supset N(f_r)$ for all r and so we can restrict our attention to one compact set. From the definition of $\int f$, we can even restrict ourselves to a compact cubic neighborhood. Then Dini's theorem (Taylor [16], page 165) tells us that the $\{f_r\}$ converge uniformly to zero and so the Riemann integrals converge to zero on each cubic set. Thus (31) holds.

Under these conditions, we can extend the integral to larger class of integrable functions, call this class L, such that $F \subset L$ and the usual theorems of integration theory, e.g., the monotone convergence theorem, the dominated convergence theorem, hold on L. We also can create a still larger class of measurable functions. A function f on M is measurable if, given any $h \in F$, $h \geq 0$, $\mathrm{mid}(h,f,-h) = (h \vee f) \wedge (-h \vee f) \wedge h$ is integrable. See Asplund and Bungart [2], Chapters 2 and 3 for details. Countable limits of measurable functions are measurable.

A sigma-field, B, of measurable sets is obtained by defining a set, A, to be measurable if its indicator function, I_A, is measurable. A measure μ is defined on B by letting $\mu(A) = \int I_A$, if $I_A \in L$, and $\mu(A) = \infty$, otherwise. We wish to show that since M has a countable base for its topology that all open sets are measurable. Let m be an arbitrary but fixed point of M and let V be a coordinate neighborhood centered at m such that $V = \{p \in M \mid |m_i(p)| < a, \ i = 1,\ldots,n\}$ where $(m_1,\ldots,m_n)$ is a chart on V and $m_i(m) = 0$, $i = 1,\ldots,n$. Let U_r be the open sphere of radius r (in the given coordinates) centered at m with $r < a$. The collection, $\{U_r\}$, at each point of M forms a base for the topology of M at the point. Since M has a countable base, a countable subcollection of all the U_r at all points is a base for the topology. If we show that the U_r are measurable, it follows that all the open sets are measurable. Since M has a countable base and is locally compact, it is sigma-compact and the collection of Borel sets is generated by the open sets. M having a countable base also implies that every Borel set is a Baire set. (Halmos [7], page 218, Theorem E).

Returning to our fixed U_r at m, consider the sequence of functions, f_t, $t \geq 1$,

$$f_t(p) = 0 \qquad\qquad p \in M - U_r$$

$$f_t(p) = (tr - td(m,p)) \wedge 1 \qquad p \in U_r$$

where $d(m,p)$ is the distance between m and p in terms of the chart $(m_1,\ldots,m_n)$. $f_t \in F$ and so $f_t \in L$ and $0 \leq f_t \leq I_r$, I_r being the indicator function of U_r. Also $f_t \leq f_{t+1}$ and $\lim_{t\to\infty} f_t = I_r$. As one can easily find a function in F that dominates the f_t for all t, we see that $\int I_r = \lim_{t\to\infty} \int f_t < \infty$ and so U_r is measurable and has finite measure. Thus all open sets are measurable. Therefore B contains all the Borel sets of M. B is in general strictly larger than the class of Borel, i.e., Baire, sets, since B is complete with respect to μ. As it is often easier to deal with the smallest σ-field containing the open sets, we shall restrict our measurable sets to be the Borel (= Baire) sets. The functions of F are measurable with respect to this smaller σ-field.

In our discussion of Sard's theorem, we gave a definition of sets of measure zero on manifolds. It is not difficult to show that if A is a Baire set, A has measure zero according to the definition if and only if $\mu(A) = 0$. Since, whenever we use the concept of measure zero with Sard's theorem, the null set is closed, it is Baire measurable and the two definitions coincide.

Another result is useful. Let M and N be two manifolds with $\varphi\colon M \longrightarrow N$ an analytic homeomorphism. Let w_M and w_N be maximal analytic differential forms on M and N respectively such that $\delta\varphi(w_N) = w_M$ and let μ_M and μ_N be the measures generated by w_M and w_N, respectively, Then, if f is integrable on N, $\int f \, d\mu_N = \int f \circ \varphi \, d\mu_M$. To show this result one need only note that if f is continuous with compact support, then

$$\int_N f \, d\mu_N = \int_N f \; w_N = \int_M (f \circ \varphi)\delta\varphi(w_N) = \int_M f \circ \varphi \; w_M = \int_M f \circ \varphi \, d\mu_M.$$

In particular, choosing f to be an indicator function, we see that $w_M = \delta\varphi(w_N)$ implies

$$\mu_M = \mu_N\varphi. \tag{33}$$

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VITA

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